Tensor Network States and Entanglement Entropy Of Stabilizer Codes

Yunqin Zheng

Princeton University

Perspectives in Topological Phase
From Condensed Matter to High Energy Physics

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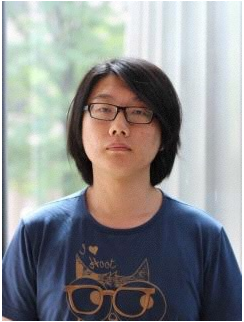

Huan He
Princeton

Andrei Bernevig
Princeton

Nicolas Regnault
ENS Paris

Motivation

Recently, Fracton models have been extensively studied.
3D Stabilizer Codes with Exotic topological order (TO).

	Mobility of Excitations	GSD[T^3]	Example
Ordinary TO	Fully mobile	$\mathcal{O}(1)$	Toric Code
Type I Fracton	Submanifold mobile	$2^{\mathcal{O}(L)}$	X-Cube
Type II Fracton	immobile	Fluctuating	Haah Code

[Haah, 2011], [Vijay, Haah, Fu, 2016]

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[Haah, 2011], [Vijay, Haah, Fu, 2016]

Entanglement behavior: Correction to the area law?
Topological/non-local correction? — Via Tensor Network State

Stabilizer Code:

- Spin model on a lattice: Spin 1/2's on links/vertices.
- Maximally Commuting Translational Invariant Hamiltonian

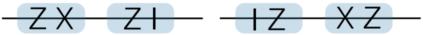
$$H = -\sum_{\mathbf{r}} \sum_{i} \mathcal{O}_i^{\mathbf{r}}$$

  i: type of interaction; $\mathbf{r}$: position;
  $\mathcal{O}_i^{\mathbf{r}}$: Hermitian, products of Pauli Z and X operators.
  $[\mathcal{O}_i^{\mathbf{r}}, \mathcal{O}_j^{\mathbf{r}'}] = 0, \forall \mathbf{r}, \mathbf{r}', i, j, \qquad (\mathcal{O}_i^{\mathbf{r}})^2 = 1, \forall \mathbf{r}, 1.$
- We will focus on the particular ground state on infinite flat space $\mathbb{R}^D$:

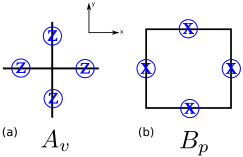
$$|GS\rangle = \prod_{\mathbf{r},i} \frac{1 + \mathcal{O}_i^{\mathbf{r}}}{2} |0\rangle$$

  which satisfies $\mathcal{O}_i^{\mathbf{r}} |GS\rangle = |GS\rangle$.

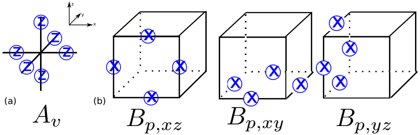
Stabilizer Code: Examples



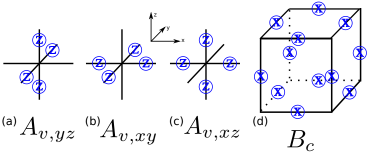
1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ SPT
(AKLT)



2D Toric Code

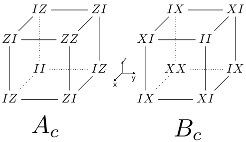


3D Toric Code



3D X Cube [Vijay, Haah, Fu, 2016]

(Type I Fracton)



Haah Code

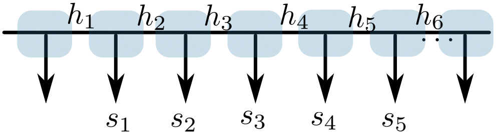
[Haah, 2011]

(Type II Fracton)

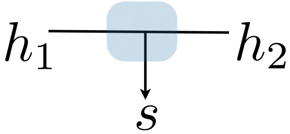
Tensor Network State

$$|\mathrm{TNS}\rangle = \sum\nolimits_{\{s\}} \mathcal{C}^{\mathbb{R}^d} \left(\ldots A^{s_1} A^{s_2} A^{s_3} \ldots \right) |\{s\}\rangle$$

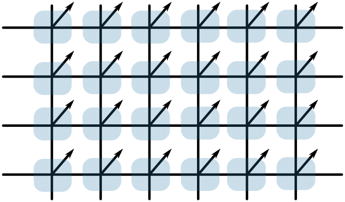
1D tensor network:
Matrix Product State



$A^s_{h_1,h_2} =$



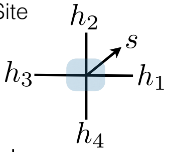
Higher Dimensional Tensor Network State

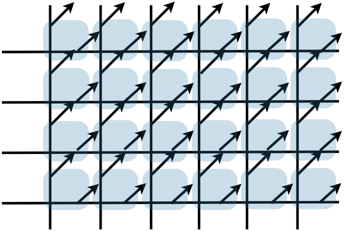


2D tensor network

1 Physical Spins per Site

$$A^{s}_{h_1,h_2,h_3,h_4} =$$





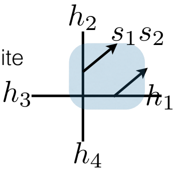
2D tensor network

Physical Spins on links

↓

2 Physical Spins per Site

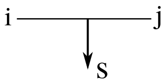
$$A^{s_1 s_2}_{h_1,h_2,h_3,h_4} =$$



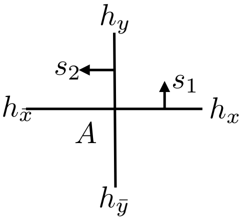
Hamiltonian $\longrightarrow |TNS\rangle$

TNS:2D Toric Code

A useful trick:
(Projector from physical index to virtual indices)

$g^{s}_{ij} =$  $= \begin{cases} 1 & s = i = j \\ 0 & \text{otherwise} \end{cases}$

$$A^{s_1 s_2}_{h_x, h_{\bar{x}}, h_y, h_{\bar{y}}} = \sum_{h'_x, h'_y} g^{s_1}_{h_x, h'_x} g^{s_2}_{h_y, h'_y} T_{h'_x, h_{\bar{x}}, h'_y, h_{\bar{y}}}$$



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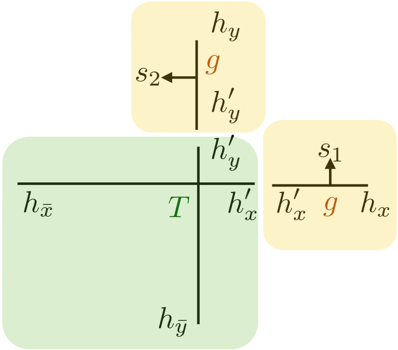
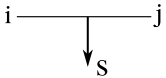


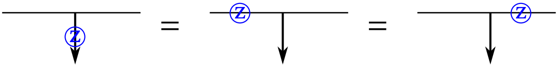
T only has virtual indices

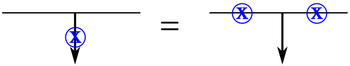
TNS: 2D Toric Code

Properties of g-tensor: $g_{ij}^{s} =$ 

$= \begin{cases} 1 & s = i = j \\ 0 & \text{otherwise} \end{cases}$

Transfer Pauli operators from physical indices to virtual indices.

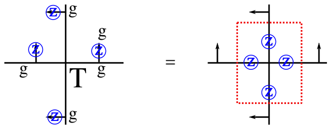
Acting with Z : 

Acting with X : 

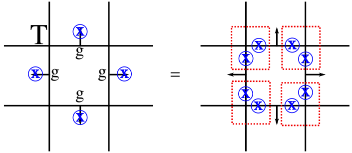
TNS: 2D Toric Code

$$A_v|TNS\rangle = |TNS\rangle$$
$$B_p|TNS\rangle = |TNS\rangle$$

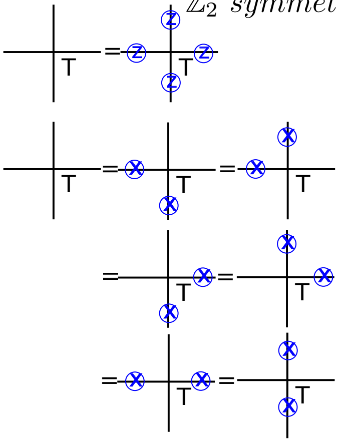
A_v



B_p



$\mathbb{Z}_2$ *symmetry*



$$T_{h_x,h_{\bar{x}},h_y,h_{\bar{y}}} = \begin{cases} 0, & \text{if } h_x + h_{\bar{x}} + h_y + h_{\bar{y}} = 1 \mod 2 \\ 1, & \text{if } h_x + h_{\bar{x}} + h_y + h_{\bar{y}} = 0 \mod 2 \end{cases}$$

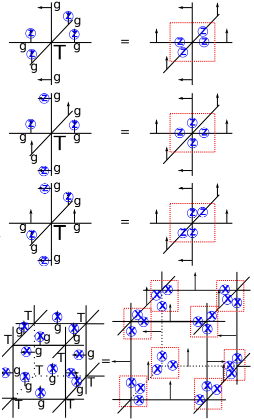
TNS: 3D X-Cube Model

$A_{v,xy}$

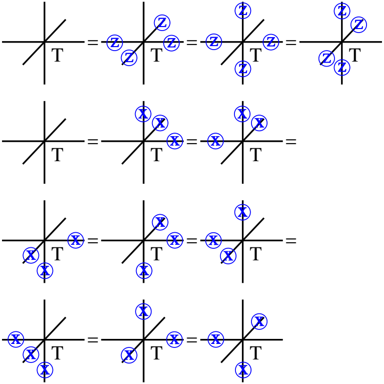
$A_{v,xz}$

$A_{v,yz}$

B_c



$\mathbb{Z}_2$ *subsystem symmetry*

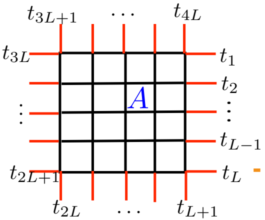


$$T_{h_x,h_{\bar{x}},h_y,h_{\bar{y}},h_z,h_{\bar{z}}} = \begin{cases} 1 & \text{if } \begin{cases} h_x + h_{\bar{x}} + h_y + h_{\bar{y}} = 0 \mod 2, \\ h_x + h_{\bar{x}} + h_z + h_{\bar{z}} = 0 \mod 2, \\ h_y + h_{\bar{y}} + h_z + h_{\bar{z}} = 0 \mod 2. \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$|TNS\rangle \longrightarrow$ **Entanglement entropy**

Entanglement Entropy via TNS: 2D Toric Code

For some entanglement cuts, the TNS naturally SVD the ground state. —> Entanglement entropy via counting



$\bar{A}$

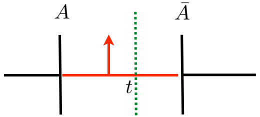
$$|\Psi\rangle = \sum_{\alpha,\beta} M_{\alpha,\beta} |\alpha\rangle_A \otimes |\beta\rangle_{\bar{A}}$$

$$|TNS\rangle = \sum_{\{t\}} |\{t\}\rangle_A \otimes |\{t\}\rangle_{\bar{A}}$$

SVD condition:

$${}_A\langle\{t'\}|\{t\}\rangle_A \propto \delta_{\{t'\},\{t\}}$$

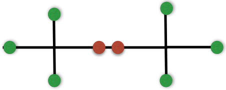
$${}_{\bar{A}}\langle\{t'\}|\{t\}\rangle_{\bar{A}} \propto \delta_{\{t'\},\{t\}}$$



Count $|\{t\}\rangle_A$

$${}_A\langle\{t\}|\{t\}\rangle_A \propto \mathcal{C}^A(...TTT...)$$

$$= 0 \; if \; \sum_t = 1 \mod 2$$



Entanglement Entropy via TNS: 2D Toric Code

- We arrive at the entanglement entropy of 2D Toric Code model:

$$\begin{aligned} S_{EE} &= \log(\textit{number of nonnull } |\{t\}\rangle_A) \\ &= \log(2^{4L-1}) \\ &= 4L\log(2) - \log(2) \end{aligned}$$

  L is the number of open virtual indices along each side of the entanglement cut.

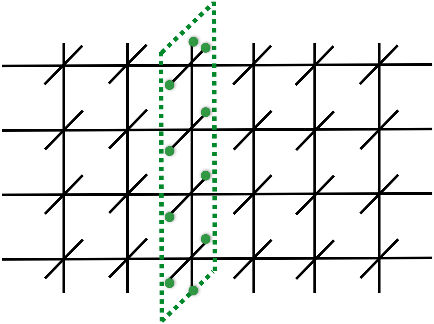
- The second term comes from nonlocal constraint of the t indices, which yields the topological entanglement entropy.
- The $3D$ Toric code similarly satisfies

$$S_{EE} = 6L^2\log(2) - \log(2)$$

Entanglement Entropy via TNS: 3D X-Cube Model

- When the region A is a cube with L open virtual indices along each direction, the entanglement entropy is

$$S_{EE} = 6L^2 \log(2) - (3L - 1)\log(2)$$
$$= [Area]\log(2) - (3L - 1)\log(2)$$



$$\sum_{i,j} t_{i,j,k} = 0 \mod 2, \quad \forall\ k$$
$$\sum_{i,k} t_{i,j,k} = 0 \mod 2, \quad \forall\ j$$
$$\sum_{j,k} t_{i,j,k} = 0 \mod 2, \quad \forall\ i$$

$$GSD[T^3] = 2^{6L-3}$$

[Vijay,Haah,Fu,2016]

Summary

- A systematic way of constructing the tensor network states for a given CSS stabilizer code (with bond dimension two) is presented. For more general stabilizer codes (with higher bond dimension and non-CSS), our way of constructing the TNS can be generalized.
- The TNS naturally leads to the ground state SVD (for certain entanglement cut). Hence the entanglement entropy can be obtained analytically from these tensor network states.
- The Entanglement entropy of the Fracton models across a topologically trivial surface have subleading order correction: $\mathcal{O}(L)$.

Thank you for your attention!

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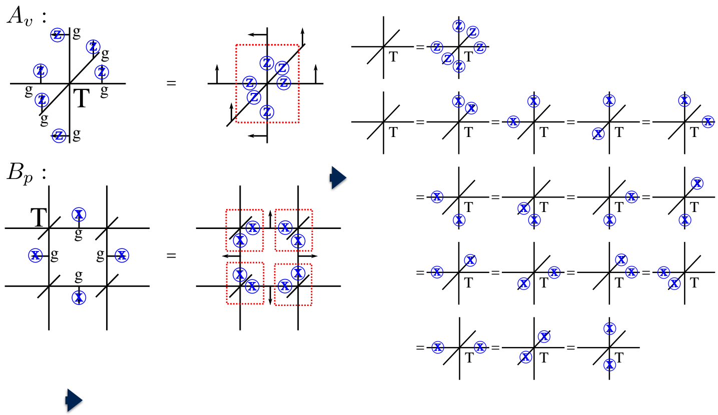
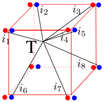
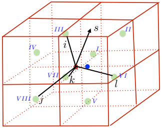
TNS: 3D Toric Code

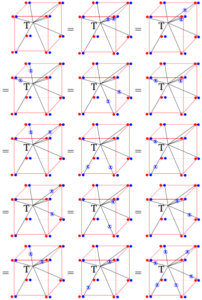


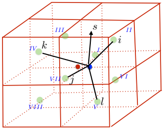
$$T_{h_x,h_{\bar{x}},h_y,h_{\bar{y}},h_z,h_{\bar{z}}} = \begin{cases} 0, & \text{if } h_x + h_{\bar{x}} + h_y + h_{\bar{y}} + h_z + h_{\bar{z}} = 1 \mod 2 \\ 1, & \text{if } h_x + h_{\bar{x}} + h_y + h_{\bar{y}} + h_z + h_{\bar{z}} = 0 \mod 2 \end{cases}$$

TNS: 3D Haah Code

T tensor: 

g^{left} tensor: 



g^{right} tensor: 

$$T_{h_1,h_2,h_3,h_4,h_5,h_6,h_7,h_8} = \begin{cases} 1 & \sum_{i=1}^{8} h_i = 0 \mod 2 \\ 0 & \sum_{i=1}^{8} h_i = 1 \mod 2 \end{cases}$$

Entanglement Entropy via TNS: 3D Haah Code

- When the region A is a cube with L open virtual indices along each direction

$$\begin{aligned} S_{EE} &= 6L^2 \log(2) - (6L-2)\log(2) \\ &= (6L^2-10)\log(2) - (6L-12)\log(2) \\ &= (Local)\log(2) - (6L-12)\log(2) \end{aligned}$$

- The cubic cut no longer yields an SVD. In the wavefunction decomposition $|TNS\rangle = \sum_{\{t\}} |\{t\}\rangle_A \otimes |\{t\}\rangle_{\bar{A}}$, there are $|\{t\}\rangle_A$ and $|\{t'\}\rangle_A$ related by the Hamiltonian terms. Hence need to identify these states by applying projection. (The entire ground state is not modified under the projection, due to the nature of the stabilizer code.)
- The *nonlocal* contribution coming the T tensor is $-(6L-12)\log(2)$.