Classification of Interacting Topological Insulators with Synthetic Dimensions

Cenke Xu

许岑珂

University of California, Santa Barbara

GORDON AND BETTY MOORE FOUNDATION

UCSB

the David & Lucile Packard FOUNDATION

Introduction

Full classification of noninteracting topological insulators and topological superconductors (Kitaev, 2009, Ryu, et.al. 2009)

Physical systems with dimensions higher than 3, was of great theoretical interests (only)…

	Symmetry					d						
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Introduction

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Viewing crystal momenta as tuning parameters in Bloch Hamiltonians, have been utilized as theoretical tools. (Qi, et.al. 2008)

$$H = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}\alpha} h^{\alpha\beta}(\mathbf{k})\, c_{\mathbf{k}\beta}$$

$$H_{\mathrm{1D}}(\theta) = \sum_{k_x} c^{\dagger}_{k_x\theta} h(k_x,\theta) c_{k_x\theta}$$

$$J_x(\theta) = G(\theta)\frac{d\theta}{dt}$$

$$\begin{aligned} G(\theta) &= -\oint \frac{dk_x}{2\pi} f_{x\theta}(k_x,\theta) \\ &= \oint \frac{dk_x}{2\pi}\left(\frac{\partial a_x}{\partial\theta} - \frac{\partial a_\theta}{\partial k_x}\right) \end{aligned}$$

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4D quantum Hall realized in the cold atom system (Lohse et al 2018) with 2 real and 2 synthetic dimensions. The "crystal momenta" of the synthetic dimensions are actually periodic slow tuning parameters.

$$V_{\mathrm{s},\mu}\sin^2(\pi\mu/d_{\mathrm{s},\mu}) + V_{\mathrm{l},\mu}\sin^2(\pi\mu/d_{\mathrm{l},\mu} - \varphi_\mu/2)$$

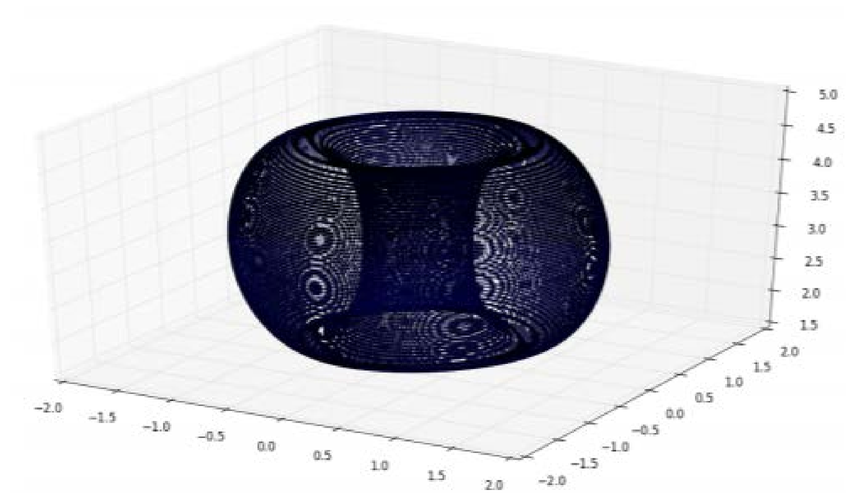
$$\nu_2 = \frac{1}{4\pi^2}\oint_{\mathrm{BZ}} \Omega^x\Omega^y \mathrm{d}k_x\mathrm{d}k_y\mathrm{d}\varphi_x\mathrm{d}\varphi_y$$

Another 4D toy TI:

Making use of the fact that $\pi_4[S^2] = \pi_4[\mathrm{Sp}(2N)/\mathrm{U}(N)] = Z_2$, one can construct a "exotic" 4D TI, with bit "complicated" crystalline protecting symmetry. Generalization of the 3D Hopf insulator.

One of the crystal momentum is a periodic tuning parameter.
The 3d edge states with zero energy:

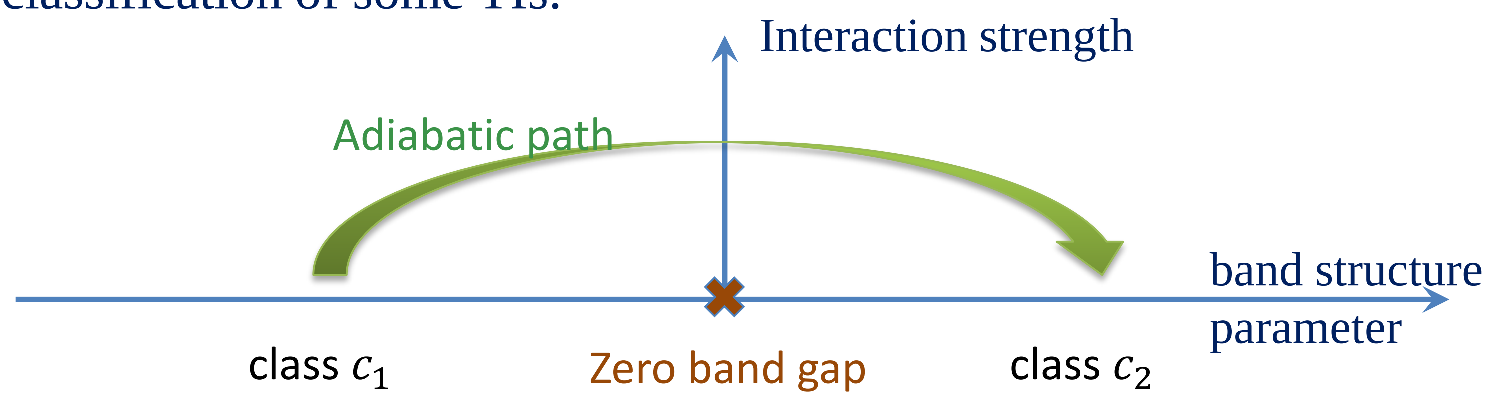
$$H(\mathbf{k}, t, m) = \vec{n}(\mathbf{k}, t, m) \cdot \vec{\sigma}.$$



Classification of Interacting Topological Insulators with Synthetic Dimensions

The goal of our work is to study interacting TIs in higher dimensions, but some of the dimensions are synthetic dimensions.

At first glance this does not look different at all from classification of interacting TIs, which has been studied a lot in the past years. We understand that spatially local interaction can collapse/reduce the classification of some TIs.



Interaction reduced classification, means two (equivalent?) things: 1, the edge states can be trivially gapped by interaction; 2 the bulk "topological" transition can be avoided with interaction.

Bulk topological transition

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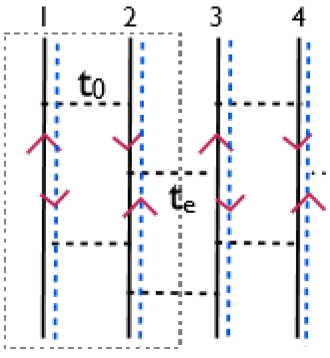

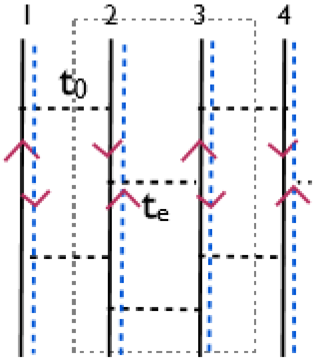

$t_o > t_e$, trivial $t_o < t_e$, topological

CC-network type of picture of the topological transition. If the trivial state and nontrivial state both have no ground state degeneracy, the topological transition can be interpreted as network of domain wall states.

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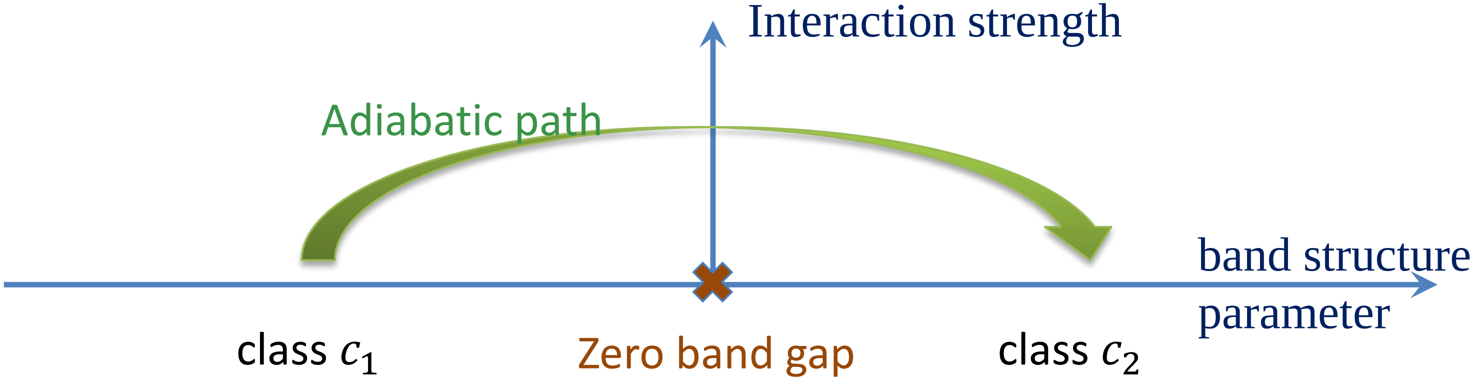


Example: $T^2 = 1$ fermions in 1D (BDI class)

Interaction reduction $\mathbb{Z} \to \mathbb{Z}_8$

Fidkowski & Kitaev, 2009

Classification of Interacting Topological Insulators with Synthetic Dimensions

But interaction in the synthetic dimensions is generically very different from ordinary local interactions: it is local in the momentum space, but nonlocal in the synthetic spatial dimensions.

Interaction in total dimension $D = d + \delta$ generally looks like this:

$$H_{int}(\vec{p}) = \int d^d x \ J_{ijkl} \psi^\dagger(\vec{x}, \vec{p})_i \psi^\dagger(\vec{x}, \vec{p})_j \psi(\vec{x}, \vec{p})_k \psi(\vec{x}, \vec{p})_l$$

It is very “nonlocal” in the synthetic dimensional subspace, can we analyze this type of interaction?

Classification of Interacting Topological Insulators with Synthetic Dimensions

We consider **TI with U(1) symmetries in $\boldsymbol{d}$ real and $\boldsymbol{\delta}$ synthetic dimensions.** Define $D = d + \delta$. Use (D, δ) for system dimensionality

Main results for two example systems:

- Non-chiral $\boldsymbol{U}(\mathbf{1}) \times \boldsymbol{Z_2}$ **symmetric TI**, two copies of class A

  Non-interacting classification: $\mathbb{Z}$ for even D and 0 for odd D

  Interaction reduced TI classification at $(\boldsymbol{D} = \mathbf{2}\boldsymbol{n}, \boldsymbol{\delta})$: $\mathbb{Z} \rightarrow \mathbb{Z}_{\mathbf{2}^{\boldsymbol{n}+\mathbf{1}-\boldsymbol{\delta}}}$

- $\boldsymbol{U}(\mathbf{1}) \times \boldsymbol{Z}_{\mathbf{2}}^{\boldsymbol{T}}$ **symmetric TI**, class AIII:

  Non-interacting classification: $\mathbb{Z}$ for odd D and 0 for even D

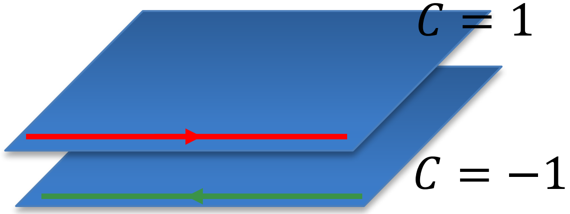
  Interaction reduced TI classification at $(\boldsymbol{D} = \mathbf{2}\boldsymbol{n} + \mathbf{1}, \boldsymbol{\delta})$: $\mathbb{Z} \rightarrow \mathbb{Z}_{\mathbf{2}^{\boldsymbol{n}+\mathbf{2}-\boldsymbol{\delta}}}$

Example 1: Non-chiral $U(1) \times Z_2$ *TI at D=2*

- **Non-interacting limit:**

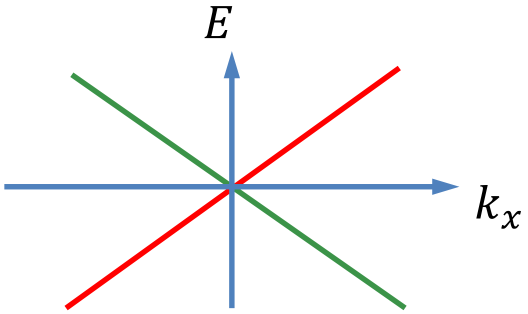
Two types of fermion with Z_2 charges $+/-$, each type has a Chern number

Total Chern number 0, $\mathbb{Z}$-classification



"Elementary state": "$C = \pm 1$ bilayer"

- **Helical boundary/edge:** $H = \int dx \left(\psi_L^\dagger \partial_x \psi_L - \psi_R^\dagger \partial_x \psi_R \right)$



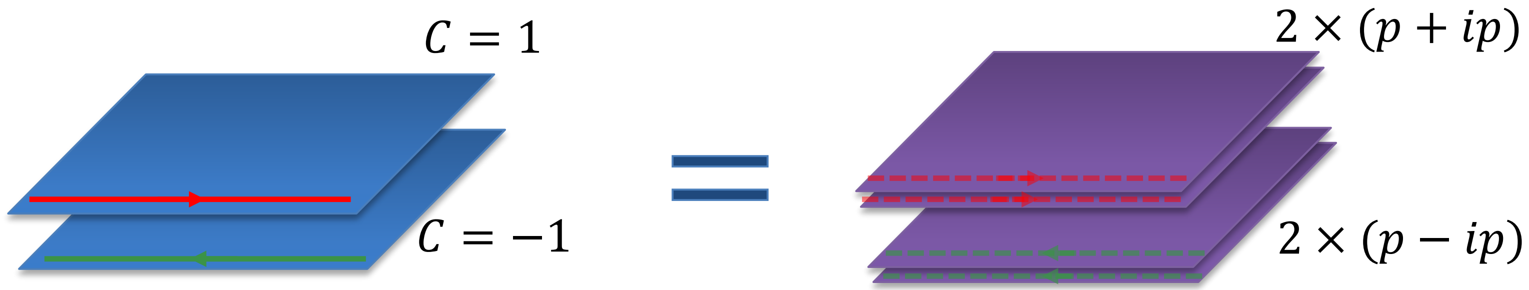
$Z_2 : \psi_L \to \psi_L \, , \psi_R \to -\psi_R$

No bilinear mass allowed:
S.C. breaks $U(1)$. Left-right mixing breaks Z_2

$\mathbb{Z}$-classification for non-interacting systems

Example 1: Non-chiral $U(1) \times Z_2$ *TI at D=2*

$D = 2$ and $\delta = 0$ (no synthetic dimensions): reduction $\mathbb{Z} \to \mathbb{Z}_4$



$U(1) = SO(2)$ rotates the 2 copies

- A single copy of "$p \pm ip$ bilayer" is a Z_2 symmetry non-chiral TSC (with $\mathbb{Z}_8$ classification). $8 \times$"$p \pm ip$ bilayer" is trivial

  Gu & Levin, 2014, and many others

- "$C = \pm 4$ bilayer" is trivial.

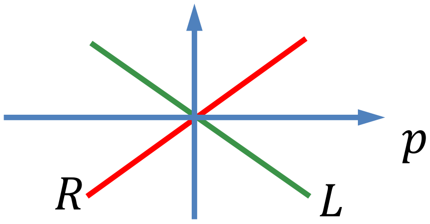
The interacting $U(1) \times Z_2$ TI at $(D, \delta) = (2, 0)$ has a $\mathbb{Z}_4$ classification.

Example 1: Non-chiral $U(1) \times Z_2$ *TI at D=2*

$D = 2$ and $\delta = 1$: reduction $\mathbb{Z} \to \mathbb{Z}_2$

- One copy of "helical" edge state with synthetic momentum p

$$H(p) = vp(\psi_L^\dagger \psi_L - \psi_R^\dagger \psi_R) + H_{int}(p)$$



Level crossing at $p = 0$ without $H_{int}(p)$.

Many-body ground states have different Z_2 charges for $p > 0$ and $p < 0$.
Level crossing is inevitable even with $H_{int}(p)$.

- Two copies: many-body ground state always have trivial Z_2 charge. Avoided crossing is allowed!

Example: $H_{int}(p) = \vec{S}_+ \cdot \vec{S}_-$

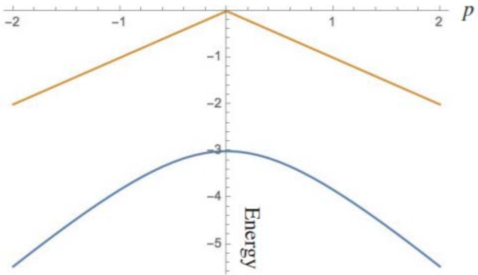
with $\vec{S}_+ \equiv (\psi_{1L}^\dagger, \psi_{2R}^\dagger)\vec{\sigma}\begin{pmatrix}\psi_{1L} \\ \psi_{2R}\end{pmatrix}, \vec{S}_- \equiv (\psi_{2L}^\dagger, \psi_{1R}^\dagger)\vec{\sigma}\begin{pmatrix}\psi_{2L} \\ \psi_{1R}\end{pmatrix}$

Example 1: Non-chiral $U(1)\times Z_2$ TI at D=2

$D=2$ and $\delta=1$: reduction $\mathbb{Z}\to\mathbb{Z}_2$

- One copy of "helical" edge state with synthetic momentum p

$$H(p)=vp\left(\psi_L^\dagger\psi_L-\psi_R^\dagger\psi_R\right)+H_{int}(p)$$



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$U(1)\times Z_2$ TI at $D=2$		
(D,δ)	(2,0)	(2,1)
Classification	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Example 2: Non-chiral $U(1) \times Z_2$ *TI at D=4*

- **Non-interacting limit:**

  Two types of fermion with Z_2 charges $+/-$, each type has a 2nd Chern number C_2

  Total 2nd Chern number 0, $\mathbb{Z}$-classification

- **Elementary state:** "$C_2 = \pm 1$ 4D Quantum Hall bilayer"

- **"3D helical boundary":** $H = \int d^3x \left(\psi_L^\dagger \vec{\sigma} \cdot i\vec{\partial}\psi_L - \psi_R^\dagger \vec{\sigma} \cdot i\vec{\partial}\psi_R \right)$

  $Z_2 : \psi_L \to \psi_L , \psi_R \to -\psi_R$ $\qquad$ $U(1) : \psi_{L/R} \to e^{i\theta}\psi_{L/R}$

  No bilinear mass allowed: S.C. breaks $U(1)$. Left-right mixing breaks Z_2

  $\mathbb{Z}$-classification for non-interacting systems

Example 2: Non-chiral $U(1)\times Z_2$ *TI at D=4*

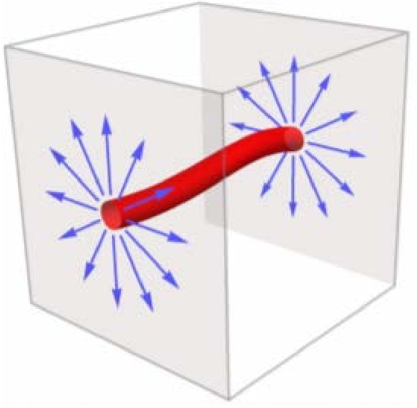
Interaction classification at $D=4, \delta=0:\mathbb{Z}\to\mathbb{Z}_8$

- Consider gapping boundary state by breaking $U(1)$ on the boundary (with S.C.) and restoring it by proliferating $U(1)$ vortex lines

- $U(1)$ vortex line carries 1D helical Majorana modes

$$H=\int dx\,(\chi_L\partial_x\chi_L-\chi_R\partial_x\chi_R)$$

with $Z_2:\chi_L\to\chi_L\,,\chi_R\to-\chi_R$, similar to the edge of the "$p\pm ip$ bilayer"



Example 2: Non-chiral $U(1) \times Z_2$ *TI at D=4*

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- $U(1)$ vortex line carries 1D helical Majorana modes

$$H = \int dx \, (\chi_L \partial_x \chi_L - \chi_R \partial_x \chi_R)$$

with $Z_2 : \chi_L \to \chi_L \, , \chi_R \to -\chi_R$, similar to the edge of the "$p \pm ip$ bilayer"

- $U(1)$ vortex line can be gapped out with 8 copies of helical Majorana modes

The interacting $U(1) \times Z_2$ TI at $(D, \delta) = (4, 0)$: $\mathbb{Z}_8$ classification

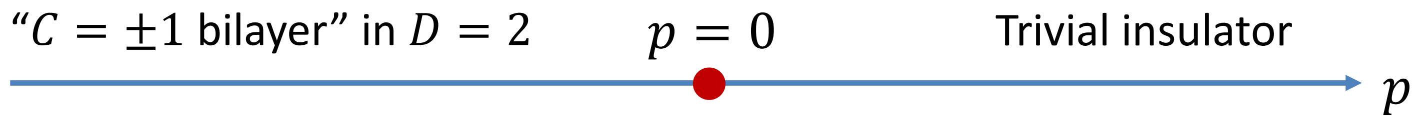
Example 2: Non-chiral $U(1) \times Z_2$ *TI at D=4*

Interaction classification at $D = 4, \delta = 1 : \mathbb{Z} \to \mathbb{Z}_4$

- Boundary state (with 1 synthetic momentum p):

$$H = \int d^2x [\, \psi_L^\dagger (\sigma^x i\partial_x + \sigma^y i\partial_y - \sigma^z p)\psi_L - \psi_R^\dagger (\sigma^x i\partial_x + \sigma^y i\partial_y - \sigma^z p)\psi_R]$$

- Same theory for $U(1) \times Z_2$ TI-to-trivial-insulator transition in $(D', \delta') = (2, 0)$ with p as the tuning parameter:



- Level crossing inevitable for one copy, but can be avoided with 4 copies, since the classification at $(D', \delta') = (2,0)$ is $\mathbb{Z}_4$.

- $H_{int}(p)$ that lifts the level crossing can be written down (for 4 copies).

Example 2: Non-chiral $U(1) \times Z_2$ *TI at D=4*

Interaction classification at $D = 4, \delta = 2 : \mathbb{Z} \rightarrow \mathbb{Z}_2$

- Boundary state (with 2 synthetic momentum p_y and p_z):

$$H = \int dx[\,\psi_L^\dagger(\sigma^x i\partial_x - \sigma^y p_y - \sigma^z p_z)\psi_L - \psi_R^\dagger(\sigma^x i\partial_x - \sigma^y p_y - \sigma^z p_z)\psi_R]$$

- $U(1) \times Z_2$ TI-trivial-insulator transition in $(\boldsymbol{D}', \boldsymbol{\delta}') = (\mathbf{2}, \mathbf{1})$:

  p_z as the tuning parameter for the phase transition and p_y still as synthetic momentum

- Level crossing inevitable for one copy, but can be avoided with 2 copies

The interacting $U(1) \times Z_2$ TI at $(D, \delta) = (4, 2)$: $\mathbb{Z}_2$ classification

Example 2: Non-chiral $U(1)\times Z_2$ *TI at D=4*

- Non-interacting classification at $D=4$: $\mathbb{Z}$

- **The interacting $U(1)\times Z_2$ TI classification at $D=4$:**

$U(1)\times Z_2$ TI at $D=4$			
(D,δ)	(4,0)	(4,1)	(4,2)
Classification	$\mathbb{Z}_8$	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Dimensional reduction

- **Compare to $D=2$:**

$U(1)\times Z_2$ TI at $D=2$		
(D,δ)	(2,0)	(2,1)
Classification	$\mathbb{Z}_4$	$\mathbb{Z}_2$

Non-chiral $U(1) \times Z_2$ *TI at general D=2n*

- Non-interacting classification at $D = 2n$: $\mathbb{Z}$

- "$(D-1)$-dimensional helical boundary state": k copies of left-handed Weyl fermions ψ_L and k copies of right-handed Weyl fermions ψ_R

- For $(D, \delta) = (2n, 0)$, interaction can trivialize edges state with $k = 2^{n+1}$ copies. **The classification at $(D, \delta) = (2n, 0)$ is $\mathbb{Z}_{2^{n+1}}$**

- Chain rule: For $\delta \geq 1$, dimensional reduction from $D = 2n$ to $D' = 2n - 2$

$$(D = 2n, \delta) \quad \sim \quad (D', \delta') = (2n-2, \delta - 1)$$

The interacting $U(1) \times Z_2$ TI at $(D = 2n, \delta)$: $\mathbb{Z}_{2^{n+1-\delta}}$ classification

Non-chiral $U(1)\times Z_2$ *TI at general D=2n*

$k=2^{n+1}$ copies of edge states trivial for $(D,\delta)=(2n,0)$

- First consider 2^n copy, write the boundary state in terms of Majorana fermions

- Couple the Majorana fermion to a $O(2n+2)$ mass vector v_i

$$Z_2:\ v_i \rightarrow -v_i \qquad\qquad U(1):\ v_1+iv_2 \rightarrow e^{i\theta}(v_1+iv_2)$$

- Fermions gapped out by v_i. Effective action of v_i is an $O(2n+2)$ WZW terms at level 1, which is identified as the boundary theory a $U(1)\times Z_2$ bosonic SPT at $D=2n$.

- Such bosonic SPT has $\mathbb{Z}_2$ classification, 2 copies of its edge state is trivial

Summary and Outlook

- Non-chiral $\boldsymbol{U(1)\times Z_2}$ **symmetric TI**, two copies of class A:

  Non-interacting classification: $\mathbb{Z}$ for even D and 0 for odd D

  Interaction reduced TI classification at $(\boldsymbol{D=2n,\delta})$: $\mathbb{Z}\rightarrow\mathbb{Z}_{\boldsymbol{2^{n+1-\delta}}}$

- Non-chiral $\boldsymbol{U(1)\times Z_2^T}$ **symmetric TI**, class AIII:

  Non-interacting classification: $\mathbb{Z}$ for odd D and 0 for even D

  Interaction reduced TI classification at $(\boldsymbol{D=2n+1,\delta})$: $\mathbb{Z}\rightarrow$ $\mathbb{Z}_{\boldsymbol{2^{n+2-\delta}}}$

Chao-Ming Jian and Cenke Xu, arXiv: 1804.03658

More exotic topological orders can potentially be realized.