Phase transitions in string nets

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Outline

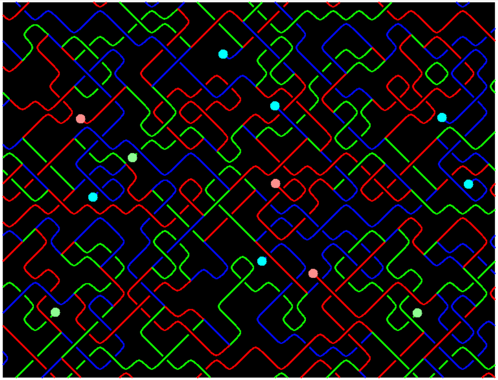
1. Topological quantum order
2. String nets with tension
3. Bound states
4. Outlook

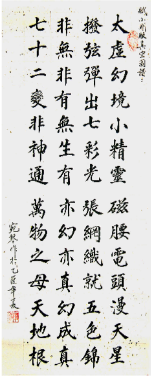
Outline

Topological quantum order

Topological quantum order in condensed matter in three dates

- 1989: High-T_c superconductors, FQHE (X.-G. Wen, F. Wilczek, and A. Zee)
- 1997: Fault-tolerant quantum computation (A. Kitaev, J. Preskill)
- 2005: String-net condensation (M. Levin and X.-G. Wen)





Courtesy of X.-G.Wen

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Main features of topologically ordered systems

2D gapped quantum systems at $T = 0$ with:

- Ground-state degeneracy depends on the system topology
- Anyonic excitations
- Long-range entanglement
- Robustness against local perturbations

  S. Bravyi, M. B. Hastings, and S. Michalakis, J. Math. Phys. **51**, 093512 (2010)

Topological quantum order

However, as early noticed...

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Condensed-matter issues

- Nature of phase transitions
- New universality classes
- Low-energy excitations

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Condensed-matter issues

- Nature of phase transitions
- New universality classes
- Low-energy excitations

Strategy

1. Start from a topological (deconfined) phase
2. Add a perturbation (string tension) $\rightarrow$ Confined phase
3. Compute the low-energy spectrum

Outline

String nets with tension

Basics

- String nets = Networks of strings
- "...topological phases originate from string-net condensation."

  M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)
- Vacuum of a topological phase = String-net condensate + Local constraints

  "Dancing patterns"

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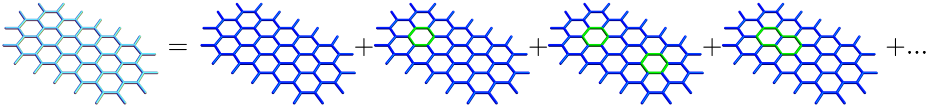
Example: the ground state of the toric code



Equal-weight superposition of all possible "loop" configurations

String nets with tension

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String-net model: input data

- Degrees of freedom are strings defined on links of a graph
- Strings are anyons (a.k.a. labels, charges, superselection sectors, particles...)
- Anyons obey fusion rules: $a \times b = \sum_c N_c^{ab} c$
- S-matrix, F-symbols, R-symbols (Unitary Modular Tensor Category)

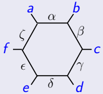
String nets with tension

Hilbert space and string-net Hamiltonian

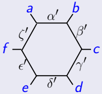
- Hilbert space: set of configurations respecting fusion rules at each vertex
- Hamiltonian defined on a trivalent graph (e.g., honeycomb lattice or ladder)

$$H_{SN} = -\sum_p B_p - \sum_v A_v$$

- B_p: projector onto the flux-free state in plaquette p

B_p 

$$= \sum_s \frac{d_s}{D^2} \sum{}' F^{a\zeta\alpha}_{s\alpha'\zeta'} F^{b\alpha\beta}_{s\beta'\alpha'} F^{c\beta\gamma}_{s\gamma'\beta'} F^{d\gamma\delta}_{s\delta'\gamma'} F^{e\delta\epsilon}_{s\epsilon'\delta'} F^{f\epsilon\zeta}_{s\zeta'\epsilon'}$$



- d_s: quantum dimension of the string s
- $D = \sqrt{\sum_s d_s^2}$: total quantum dimension
- Remark: $\frac{d_s}{D^2} = S_{11} S_{1s}$ but nontrivial Frobenius-Schur indicator matters !

M. Levin and X.-G. Wen, Phys. Rev. B **71**, 045110 (2005)

String nets with tension

The flux picture

- Ground state: $B_p|\psi_0\rangle = +|\psi_0\rangle \rightarrow$ No flux in plaquette p
- Excited states: $B_p|\psi_{\text{exc.}}\rangle = 0 \quad \rightarrow$ Non-trivial flux in plaquette p

  F. J. Burnell and S. H. Simon, Ann. Phys. **325**, 2550 (2010); S. H. Simon and P. Fendley, J. Phys. A **46**, 105002 (2013)

- Excited states are closed flux lines piercing plaquettes
- Doubled achiral topological deconfined phase

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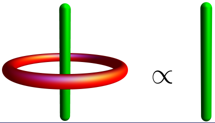
How to measure a flux ?

- Probing a flux $\Leftrightarrow$ Insert a loop around a flux (Aharonov-Bohm effect)
- Projector onto a given flux $\Leftrightarrow$ Combination of loop operators



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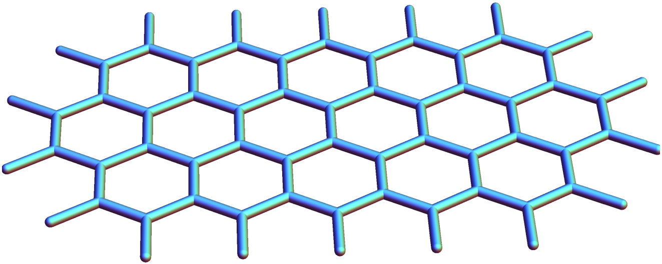

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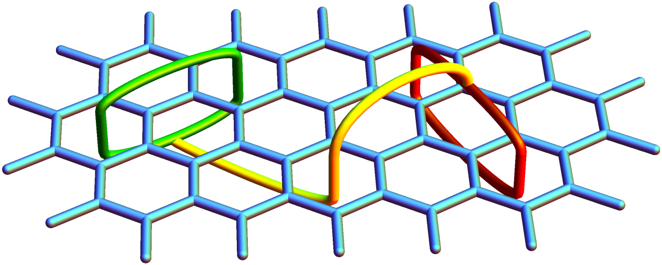

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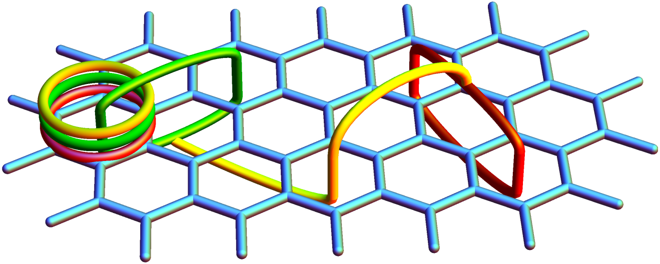
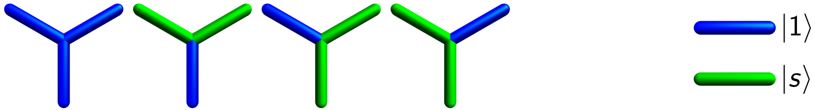


String nets with tension

The simplest Abelian case: the semion theory

- Two strings: $\{1, s\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times s = s$, $s \times s = 1$

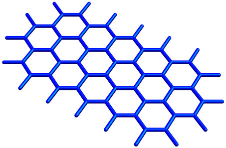


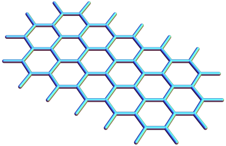
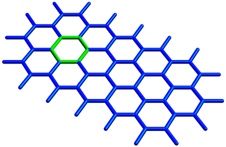
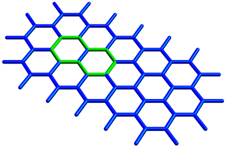
Hilbert space for any graph with N_{v} trivalent vertices

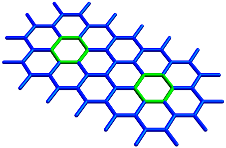
- Dim $\mathcal{H} = 2^{\frac{N_{\text{v}}}{2}+1}$

String nets with tension

Ground state of the semion theory (loop theory)

1  $+$

 $\propto \; (-1)$  $+(-1)$  $+\ldots$

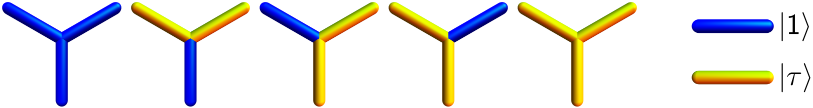
1  $+\ldots$

Weight of a configuration $= (-1)^{\#\mathrm{loops}}$

String nets with tension

The simplest non-Abelian case: the Fibonacci theory

- Two strings: $\{1, \tau\}$
- Fusion rules: $1 \times 1 = 1$, $1 \times \tau = \tau$, $\tau \times \tau = 1 + \tau$

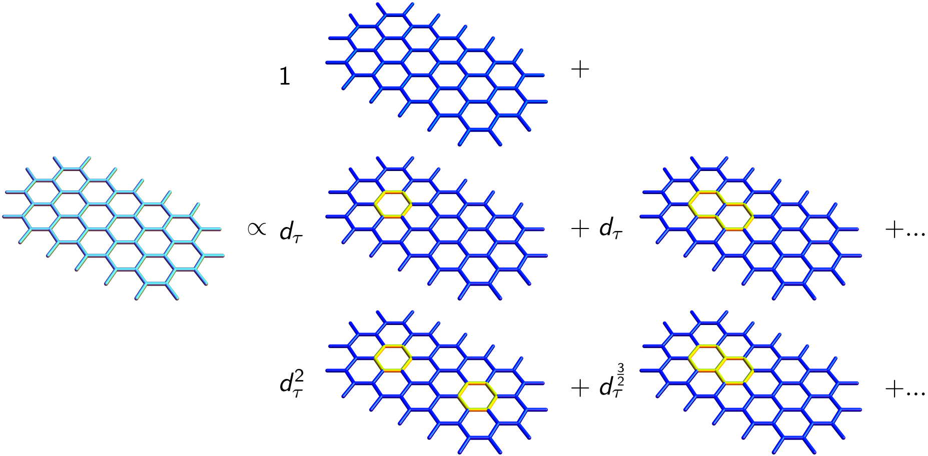


Hilbert space for any graph with N_v trivalent vertices

- Dim $\mathcal{H} = (1+\varphi^2)^{\frac{N_\text{v}}{2}} + (1+\varphi^{-2})^{\frac{N_\text{v}}{2}}$ with $\varphi = \frac{1+\sqrt{5}}{2}$ (golden ratio)

String nets with tension

Ground state of the Fibonacci theory



$d_\tau = \varphi$ is the quantum dimension of the string τ

String nets with tension

Strategy

1. Start from a topological (deconfined) phase
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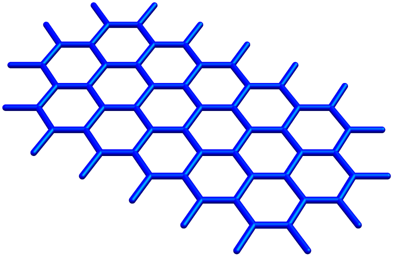
3. Compute the low-energy spectrum

String nets with tension

A simple local perturbation $\sim$ string tension

- Projector onto the flux-free state in the links: $V = \sum_\ell \delta_{\ell,1} = \sum_\ell C_\ell$
- Diagonal in the link basis but nontrivial in the flux basis
- $[B_p, C_\ell] \neq 0 \rightarrow V$ creates fluxes in plaquettes
- V generates dynamics and interactions between fluxes



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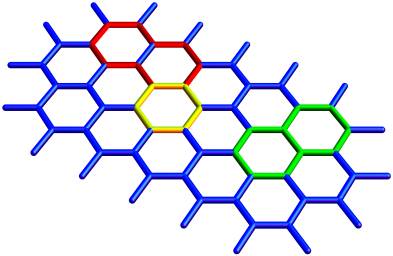

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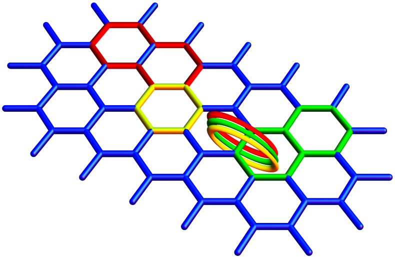

String nets with tension

The model

$$H = -\cos\theta \sum_p B_p - \sin\theta \sum_\ell C_\ell$$

- $B_p =$ Projector onto the flux-free state in the plaquette p
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The ladder

- Exact mapping onto transverse-field D^2-state Potts model (1+1)
- $D \leqslant 2$: 2^{nd}-order transitions described by CFT

  C. Gils, S. Trebst, A. Kitaev, A. W. W. Ludwig, M. Troyer, and Z. Wang, Nat. Phys. **5**, 834 (2009)

  C. Gils, J. Stat. Mech. P07019 (2009)

  E. Ardonne, J. Gukelberger, A. W.W. Ludwig, S. Trebst, M. Troyer, New J. Phys. 13, 045006 (2011)
- $D > 2$: 1^{st}-order transitions

  M. D. Schulz, S. Dusuel, and J. Vidal, Phys. Rev. B **91**, 155110 (2015)

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The honeycomb lattice

- Abelian theories $\sim$ transverse-field D^2-state Potts model (2+1)

  F. J. Burnell, S. H. Simon, J. K. Slingerland, Phys. Rev. B **84**, 125434 (2011)

- Non-Abelian theories ?
- Tools: High-order perturbation theory, mean field, exact diagonalizations

  M. D. Schulz, S. Dusuel, K. P. Schmidt, and J. Vidal, Phys. Rev. Lett. **110**, 147203 (2013)

  M. D. Schulz, S. Dusuel, G. Misguich, K. P. Schmidt, and J. Vidal, Phys. Rev. B **89**, 201103(R) (2014)

  S. Dusuel, and J. Vidal, Phys. Rev. B **91**, 155110 (2015)

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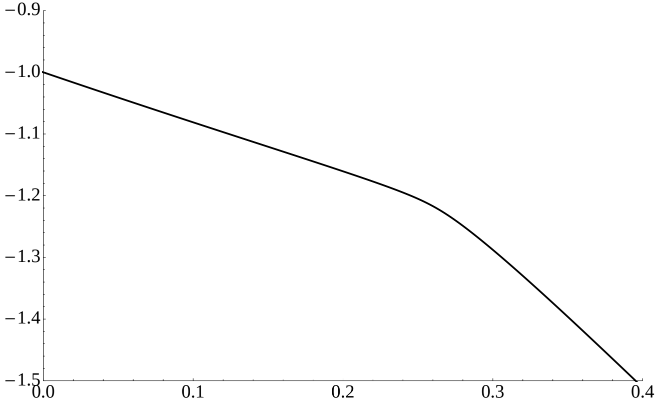
Non-Abelian theories with N strings on a genus g surface

- $\theta = 0$: No flux in the plaquette (degeneracy genus- and theory-dependent)
- $\theta = \pi/2$: No flux in the links (degeneracy $= 1$)
- $\theta = \pi$: One flux in each plaquette (degeneracy $= \infty$)
- $\theta = 3\pi/2$: One flux in each link (degeneracy $= \infty$ for $N > 2$)

Fibonacci theory $s = 1, \tau$

Ground-state energy per plaquette *vs* θ

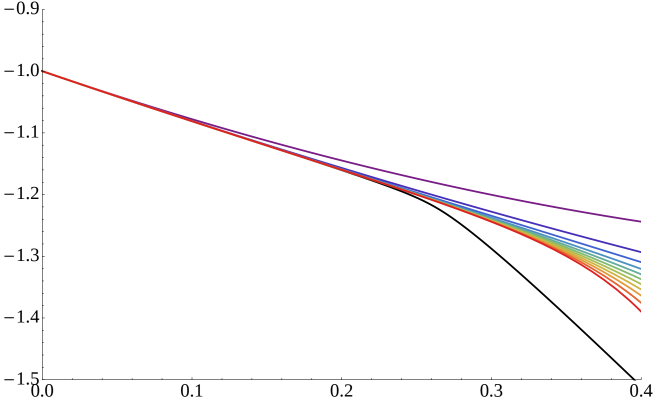
ED with $N_p = 13$ on a torus



Fibonacci theory $s = 1, \tau$

Ground-state energy per plaquette *vs* θ

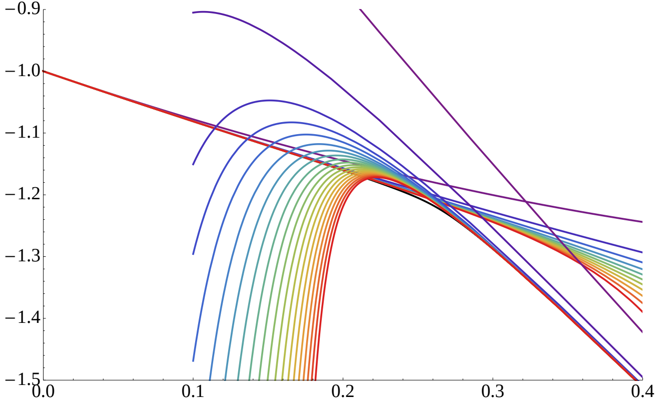
ED for $N_p = 13$ on a torus + Series near $\theta = 0$ (order 1 to 11)



Fibonacci theory $s = 1, \tau$

Ground-state energy per plaquette *vs* θ

ED with $N_p = 13$ on a torus + Series near $\theta = 0$ (order 1 to 11) + Series near $\theta = \pi/2$ (order 1 to 20)



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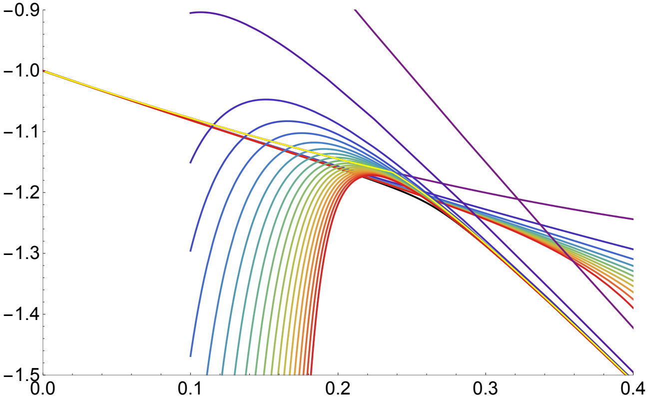
Ground-state energy per plaquette *vs* θ

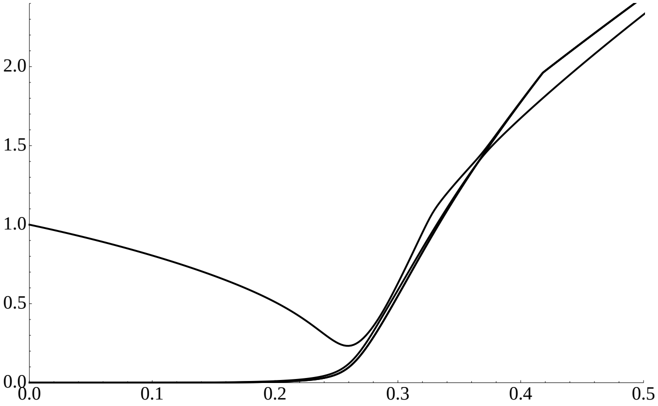
ED with $N_p = 13$ on a torus + Series near $\theta = 0$ (order 1 to 11) + Series near $\theta = \pi/2$ (order 1 to 20)+ Mean field



Fibonacci theory $s = 1, \tau$

Excitation spectrum *vs* θ

ED with $N_p = 13$ on a torus



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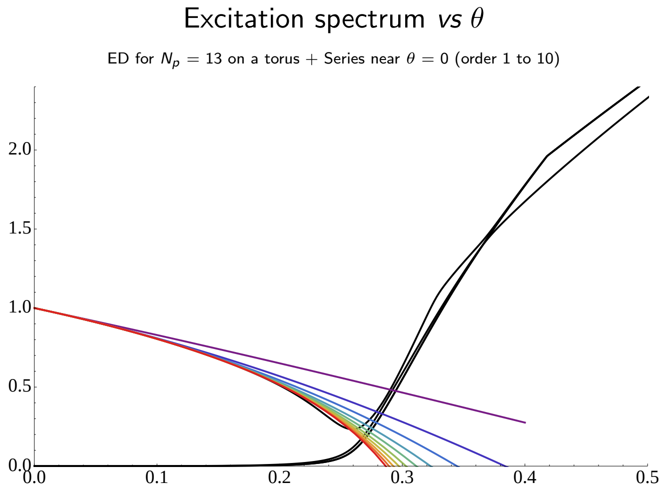

Fibonacci theory $s = 1, \tau$

Excitation spectrum *vs* θ

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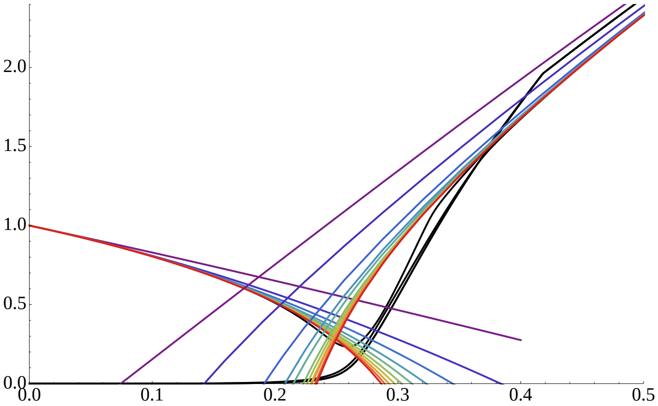
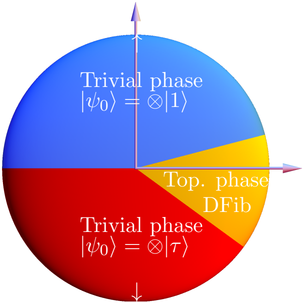


String nets with tension

Phase diagrams for different theories

Fibonacci



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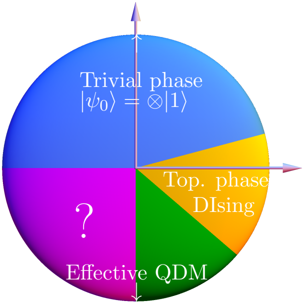
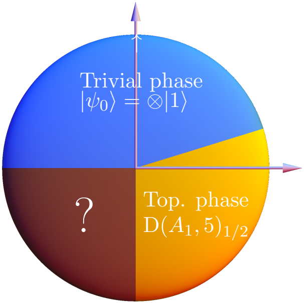


$s = 1, \sigma, \psi$

SU(2)$_5$



$s = 1, \alpha, \beta$

Nature of transitions still unknown !

Outline

Bound states

Bound states and quantum phase transitions

- No bound state $\Rightarrow$ 2^{nd}-order transition ? Yes !
  Ex: $2 \leqslant q \leqslant 3$-state Potts model in $(1+1)$ dimensions
- Bound states $\Rightarrow$ 1^{st}-order transition ? No !
  Ex: $3 < q \leqslant 4$-state Potts model in $(1+1)$ dimensions

  *P. Dorey, A. Pocklington, and R. Tateo, Nucl. Phys. B **661**, 425 (2003)
- 1^{st}-order transition $\Rightarrow$ Bound states ? Yes ! (in general...)
  Ex: $4 < q$-state Potts model in $(1+1)$ dimensions

Bound states

Bound states and quantum phase transitions

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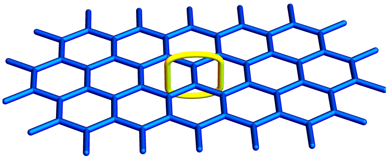
Role of the string tension

- Dynamics + interaction between fluxes
- Anyonic bound states ?

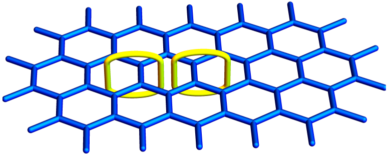
  M. D. Schulz, S. Dusuel, and J. Vidal, Phys. Rev. B **94**, 205102 (2016)

Two-flux excited state



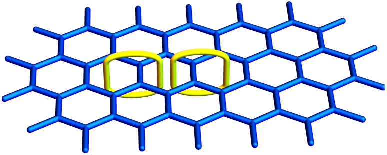
Bound states

Two-flux excited state + Perturbation

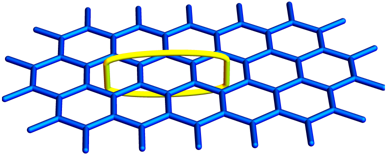


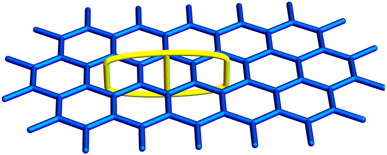
Bound states

Two-flux excited states + Perturbation



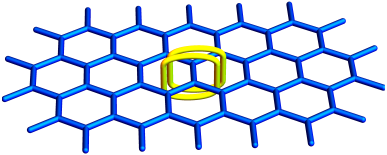
Flux hopping ↙ ↘ Flux creation



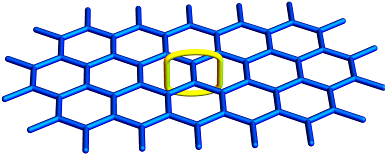


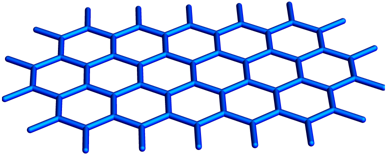
Fibonacci fusion rules: $\tau \times \tau = 1 + \tau$

Two-flux excited state + Perturbation



Interaction ↙ ↘ Flux destruction





Fibonacci fusion rules: $\tau \times \tau = 1 + \tau$

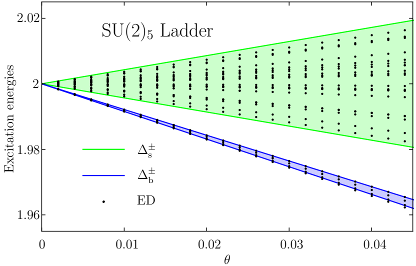
Bound states

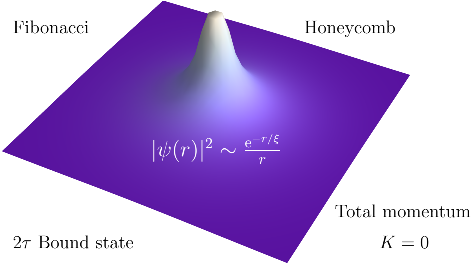
Results valid for any (modular) theory

Perturbative approach in the limit $\theta \ll 1$:

- Ladder : 1 Bound state for $3 < D^2$
- Honeycomb: 1 Bound state for $2 < D^2 < D_c^2$
  3 Bound states for $D_c^2 < D^2$
- "Critical" total quantum dimension $D_c \simeq 3.87145 + \mathcal{O}(\theta^2)$

M. D. Schulz, S. Dusuel, and J. Vidal, Phys. Rev. B **94**, 205102 (2016)





Outlook

Take-home messages and perspectives

- String-nets can describe many topological phases
- Confinement-deconfinement transitions are induced by string tensions
- Tensor networks approach (perturbative PEPS)
- Transitions between compatible topological phases (same fusion rules)
- New universality classes in two dimensions to be unveiled