Topology and entanglement detected by fermionic partial transpose

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Collaborators and References

- Collaborators: Hassan Shapourian (U Chicago) and Ken Shiozaki (RIKEN)

- Papers:
  - “Partial time-reversal transformation and entanglement negativity in fermionic systems”, arXiv:1611.07536
  - "Detection of symmetry-protected topological phases in fermionic many-body systems", arXiv:1607.03896
  - “Many-body topological invariants for fermionic short-range entangled topological phases protected by antiunitary symmetries”, arXiv:1710.01886
  - "Entanglement negativity of fermions: monotonicity, separability criterion and classification of few-mode states ", arXiv:1804.08637

Summary

- Construction of partial transpose operation for fermionic systems; ("femrionic partial transpose")
- Fermionic partial transpose can be used (i) to detect fermionic quantum entanglement by entanglement negativity.
- and (ii) to detect topological invariants of fermionic topological phases protected by time-reversal (and others).

Outline

1. Partial transpose (in bosonic systems)
    - Useful operation to diagnose entanglement,
    - and topological phases
2. Fermionic partial transpose
    - Issues
    - Our construction
    - Motivation behind our construction
    - LOCC monotone
3. Applications
    - 2d Topological insulators
    - Fermi surface, strong randomness

Partial transpose: bosonic case

- Definition: for the density matrix $\rho_{A_1 \cup A_2}$,

$$\langle e_i^{(1)} e_j^{(2)} | \rho_{A_1 \cup A_2}^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho_{A_1 \cup A_2} | e_k^{(1)} e_j^{(2)} \rangle$$

  where $|e_i^{(1,2)}\rangle$ is the basis of $\mathcal{H}_{A_1, A_2}$.

- Detecting quantum correlation coming from "off-diagonal" parts:
  Entanglement negativity and logarithmic negativity:

$$\frac{1}{2}(\mathrm{Tr}\,|\rho_A^{T_2}| - 1), \quad \mathcal{E}_A = \log\,\mathrm{Tr}\,|\rho_A^{T_2}|$$

  [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

- Partial transpose $\simeq$ partial time-reversal

$$H^* = H^T$$

  Useful for detecting topological phases with time-reversal symmetry.

Partial transpose and quantum entanglement

- Bell pair: $|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|01\rangle - |10\rangle\right]$

$$\rho = |\Psi\rangle\langle\Psi| = \frac{1}{2}\left[|01\rangle\langle 01| + |10\rangle\langle 10| - |01\rangle\langle 10| - |10\rangle\langle 01|\right]$$

How do we quantify quantum entanglement?

- Partial transpose:

$$\rho^{T_2} = \frac{1}{2}\left[|01\rangle\langle 01| + |10\rangle\langle 10| - \underline{|00\rangle\langle 11|} - \underline{|11\rangle\langle 00|}\right]$$

- Entangled states are badly affected by partial transpose: Negative eigenvalues: $Spec(\rho^{T_2}) = \{1/2, 1/2, 1/2, -1/2\}$.
- C.f. For a classical state:

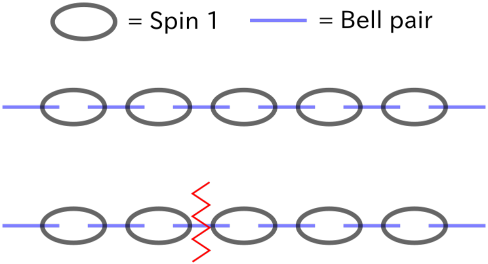
$$\rho = \frac{1}{2}\left[|00\rangle\langle 00| + |11\rangle\langle 11|\right] = \rho^{T_2}$$

Partial transpose and Entanglement negativity

- How to quantify quantum entanglement between A_1 and A_2 when $\rho_{A_1 \cup A_2}$ is *mixed* ? E.g., finite temperature, $A_{1,2}$ is a part of bigger system.
- The entanglement entropy is an entanglement measure only for pure states.
- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose*, can extract quantum correlations only. [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]
- The logarithmic negativity is not convex but an entanglement monotone. [Plenio (2005)]

Partial transpose and topological phases: e.g., Haldane phase

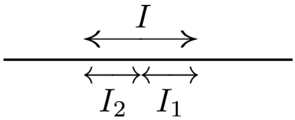
- Spin 1/2 edge state:



- Quantum anomaly: edge states are not invariant under $SO(3)$ rotation, but pick up a phase (-1)
- Haldane state = collection of Bell pairs

Partial transpose and topological invariant

- Partial transpose can be used to construct/define topological invariants of bosonic topological phases [Pollmann-Turner]



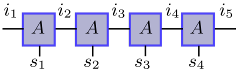
- Step 1: The reduced density matrix for an interval I, $\rho_I := \mathrm{Tr}_{\bar{I}}|\Psi\rangle\langle\Psi|$.
- Step 2: Bipartition I into two *adjacent* intervals, $I = I_1 \cup I_2$.
- Step 3: Take *partial time-reversal* acting only on I_1; $\rho_I \longrightarrow \rho_I^{T_1}$.
- Step 4: The invariant is given by the phase of:

$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}],$$

and ± 1. C.f. Negativity: $\mathrm{Tr}\,|\rho_I^{T_1}|$

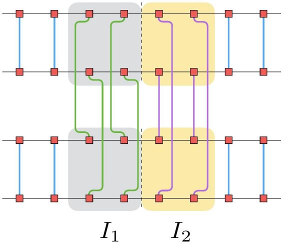
- Matrix product state representation:
  - Wave function;

$$\Psi(s_1, s_2, \cdots) = \sum_{\{i_n=1,\cdots\}} A^{s_1}_{i_1 i_2} A^{s_2}_{i_2 i_3} A^{s_3}_{i_3 i_4} \cdots \quad s_a = \uparrow, \downarrow$$

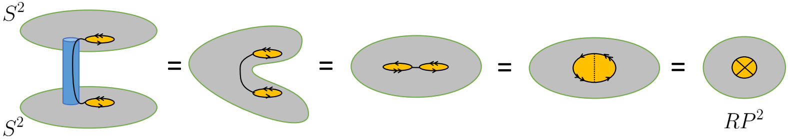


  - Topological invariant:

$$Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}]$$



- The invariant "simulates" the path integral on real projective plane $\mathbb{R}P^2$: [Shiozaki-Ryu (16)]



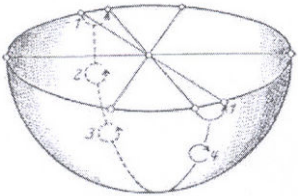


Abb. 9.

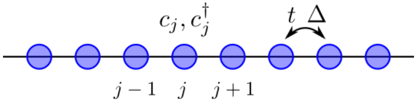
Summary so far

- Partial transpose is useful to detect entanglement and topological properties of many-body states.
- Partial transpose can generate spacetimes which are unorientable.
- How about fermion systems? E.g., the Kitaev chain

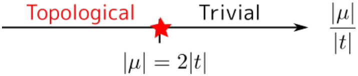
The Kitaev chain

- The Kitaev chain

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



- Phase diagram: there are only two phases:

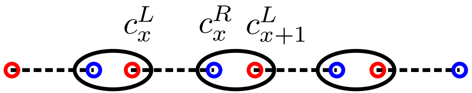


- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

Majorana dimers

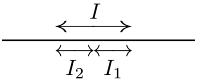
- Fractionalizing an electron into two Majoranas:

$$c_x = c_x^L + ic_x^R, \quad c_x^\dagger = c_x^L - ic_x^R.$$

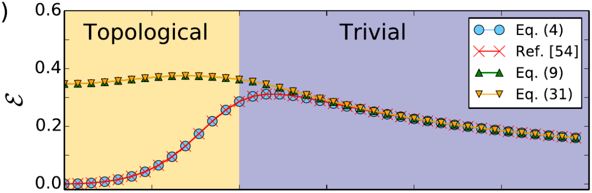


Issues in fermionic systems (1)

- Consider log negativity $\mathcal{E}$ for two adjacent intervals of equal length. $(L = 4\ell = 8)$



(a)



- Vertical axis: μ/t ranging from 0 to 6.
- (Blue circles and Red corsses) is computed by Jordan-Wigner + bosonic partial transpose
- Log negativity fails to capture Majorana dimers.

Issues in fermionic systems (2)

- Partial transpose of bosonic Gaussian states is still Gaussian; easy to compute by using the correlation matrix
- Partial transpose of fermionic Gaussian states are not Gaussian
  - ρ^{T_1} can be written in terms of two Gaussian operators $O_\pm$:

$$\rho^{T_1} = \frac{1-i}{2} O_+ + \frac{1+i}{2} O_-$$

  - Negativity estimators/bounds using $\mathrm{Tr}\left[\sqrt{O_+ O_-}\right]$ [Herzog-Y. Wang (16), Eisert-Eisler-Zimborás (16)]
  - Spin structures: [Coser-Tonni-Calabrese, Herzog-Wang]

Topological/geometrical insight into partial transpose

- Lesson from the Haldane phase example: partial transpose can change the topology of spacetime: quantum field theory on an unoriented spacetime [Pollmann-Turner, Calabrese-Cardy-Tonni, Shiozaki-SR]
- The relevant TQFT are invertible, fermionic and defined on unoriented spacetime ("Pin" TQFT) [Kapustin, Hsieh-Cho-Sule-SR-Leigh, Kapustin-Thorngren-Turzillo-Wang, Hsieh-Cho-SR, Witten, Freed-Hopkins, Metlitski, Barkeshli-Bonderson-Jian-Cheng-Walker, Yonekura-Tachikawa, and many others]
- For Majorana fermions, we should also be able to give a topological interpretation for partial transpose.
- We use topological quantum field theory as a guide to search for a proper definition of partial transpose for fermions.

Partial transpose for fermions – our definition

[Shiozaki-Shapourian-SR (16)]

- Fermion operator algebra does not trivially factorize for $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$.
- Expand the density matrix in terms of Majorana fermions:

$$\rho_A = const. + \sum_{p_{1,2}} \rho_{p_1 p_2} c_{p_1} c_{p_2} + \sum_{p_{1,\dots,4}} \rho_{p_1 p_2 p_3 p_4} c_{p_1} c_{p_2} c_{p_3} c_{p_4} + \cdots$$

- Group them in terms of subregions:

$$\rho_A = \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} \underbrace{c^{A_1}_{p_1} \cdots c^{A_1}_{p_m}}_{\in A_1} \underbrace{c^{A_2}_{q_1} \cdots c^{A_2}_{q_n}}_{\in A_2}$$

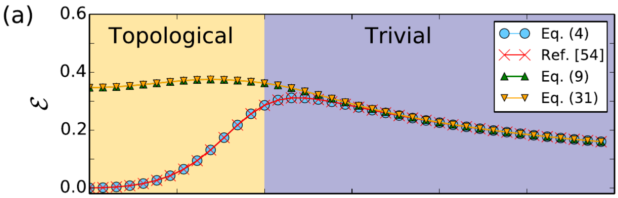
- Define partial transpose by $\rho_{p,q} \to \rho_{p,q} i^m$:

$$\rho_A^{T_1} = \sum_{m,n}^{m+n=even} \sum_{\{p_i, q_j\}} \rho_{p_i, q_j} i^m c^{A_1}_{p_1} \cdots c^{A_1}_{p_m} c^{A_2}_{q_1} \cdots c^{A_2}_{q_n}$$

- C.f. fermionic matrix product states perspective [Bultinck et al]
- Gaussian states stay Gaussian under our partial transpose

Comparison with previous definitions

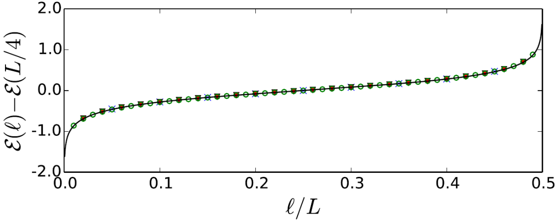
[Shiozaki-Shapourian-SR (16)]



- (Blue circles and Red crosses): Old (bosonic) definition
- (Green triangles and Orange triangles) Our definition;
- At critical point: agrees with CFT prediction by Calabrese-Cardy-Tonni.

Critical point

- The logarithmic negativity for two adjacent intervals of equal length ℓ at the critical point (the SSH model).

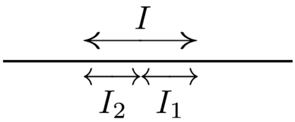


- The numerical result using the free fermion formula (points) with $L = 40$-400 agrees with the CFT result (solid line). [Calabrese-Cardy-Tonni]

$$\mathcal{E} = \frac{c}{4} \ln \tan \frac{\pi \ell}{L}$$

- Analytical derivation by using the replica method + Fisher-Hartwig.

Topological invariant for TRS Majorana chain



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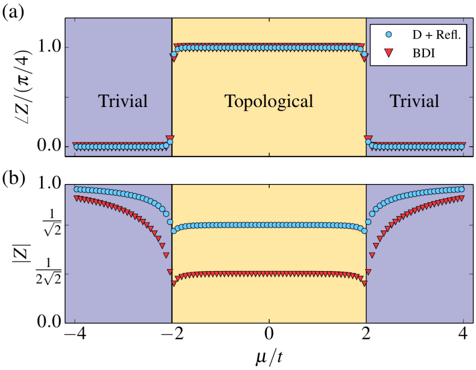
- Step 4: The invariant is given by the phase of: $Z = \mathrm{Tr}[\rho_I \rho_I^{T_1}]$
- This quantity should correspond, in the continuum limit, the partition function of the Kitaev chain on the real projective plane.

Numerics

- Numerics



- The phase of Z is quantized to the 8th root of unity. Consistent with $\mathbb{Z}_8$ classification: [Fidkowski-Kitaev(10)]

Monotoncity under LOCC

- For bosonic systems, the entanglement negativity is entanglement monotone under LOCC (local quantum operations and classical communications).

$$\rho \longrightarrow (A \otimes B)\rho(A \otimes B)^\dagger$$
$$\text{or} \quad \rho \longrightarrow (A^i \otimes 1)(1 \otimes B_i)\rho(1 \otimes B_i)^\dagger (A^i \otimes 1)^\dagger$$
$$\text{but not} \quad \rho \longrightarrow K_{AB}\, \rho\, K_{AB}^\dagger.$$

- I.e., what cannot be generated by LOCC = "quantum entanglement".

Monotoncity under LOCC

- von-Neumann Entanglement entropy decreases monotonically at $T = 0$, but *not* at $T > 0$.
- (Ordinary) negativity decreases monotonically under LOCC.
- We have introduced fermionic version of partial transpose, and negativity, but is it a good entanglement measure? Is it monotone under LOCC?
- Proved fermionic entanglement negativity is monotone, if LOCC are taken to be fermion number parity preserving. [Shapourian-SR (18)]

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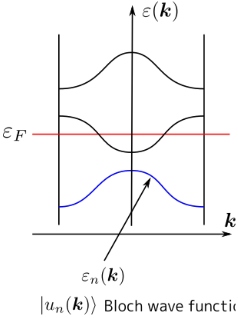
Application: 2d time-reversal symmetric topological insulators

[Shiozaki-Shapourian-SR (17)]

- Fermionic topological phase protected by time-reversal and charge $U(1)$.
- Conventionally discussed in terms of free fermion band theory

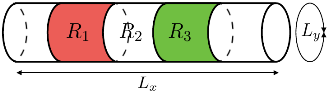


- Fully many-body formulation is now possible with fermionic partial transport.

Many-body $\mathbb{Z}_2$ topological invariant

[Shiozaki-Shapourian-SR (17)]

- Setup:



- Formula: (T_1 = fermionic partial transpose)

$$Z = \mathrm{Tr}_{R_1 \cup R_3}\left[\rho^{+}_{R_1\cup R_3} C_T^{I_1}[\rho^{-}_{R_1\cup R_3}]^{\mathsf{T}_1}[C_T^{I_1}]^{\dagger}\right],$$

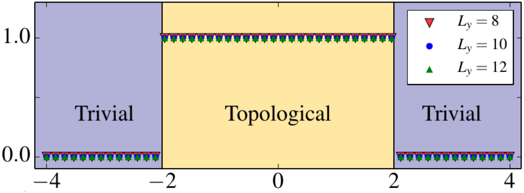
$$\rho^{\pm}_{R_1\cup R_3} = \mathrm{Tr}_{\overline{R_1\cup R_3}}\Big[\underbrace{e^{\pm\sum_{\mathbf{r}\in R_2}\frac{2\pi i y}{L_y}n(\mathbf{r})}}_{\text{partial } U(1) \text{ twist}}|GS\rangle\langle GS|\Big]$$

$C_T \sim$ spin flip unitary

- Z is the partition function on Klein bottle $\times\, S^1$ with flux.

Many-body $\mathbb{Z}_2$ topological invariant

- Numerics on a lattice:



Application: Fermi surface at finite T

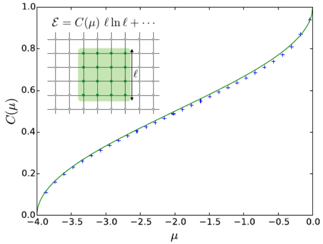
- Renyi entanglement entropy:

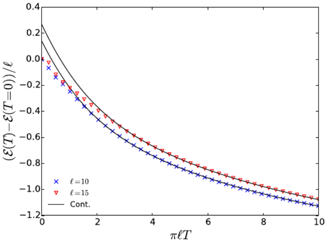
$$S_n = \frac{n+1}{6n} C_2 \cdot \ell \ln \left| \frac{\beta}{\pi a_0} \sinh \frac{\pi \ell}{\beta} \right|$$

$$\text{where} \quad C_2 = \frac{1}{8\pi} \int_{\partial\Omega} \int_{\partial\Gamma} dS_k dS_x |n_x \cdot n_k|$$

- Negativity:

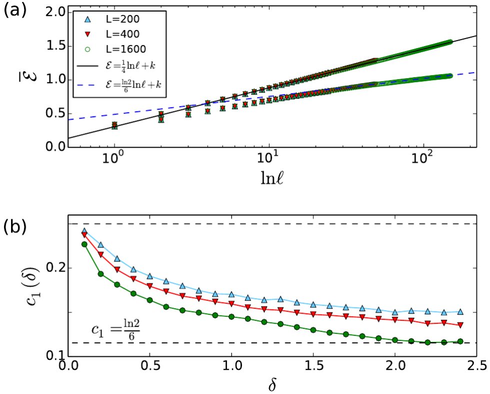
$$\mathcal{E} = C_2 \cdot \frac{\ell}{2} \left[\ln \left(\frac{\beta}{\pi a_0} \sinh \frac{\pi \ell}{\beta} \right) - \frac{\pi \ell}{\beta} \right]$$





- No sudden death

Application: Randomness

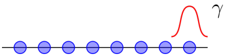


-
- The result is consistent with the strong randomness RG.

Summary

- Based on the topological field theory intuition, we introduced partial transpose (time-reversal) for fermionic systems.
- The (log) negativity using the fermionic partial transpose can capture the formation of Majorana dimers in the Kitaev chain.
- Similar constructions of *many-body* topological invariants for other fermionic SPT phases; e.g. $\mathbb{Z}_{16}$ invariant for (3+1)d topological superconductors, $\mathbb{Z}_2$ time-reversal symmetric topological insulators.
- Partial transpose of fermionic Gaussian states are Gaussian, and hence easy to compute.

the Kitaev chain in zero correlation limit



- The reduced density matrix

$$\rho_A = \frac{1}{4}\left[(1+f_1^\dagger f_2^\dagger)|0\rangle\langle 0|(1+f_2 f_1) + (f_1^\dagger + f_2^\dagger)|0\rangle\langle 0|(f_1+f_2)\right]$$

- $$\rho_A = \frac{1}{4}\begin{bmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & 1 & 1 & \\ 1 & & & 1 \end{bmatrix} \Longrightarrow \rho_A^{R_1} = \frac{1}{4}\begin{bmatrix} 1 & & & i \\ & 1 & i & \\ & i & 1 & \\ i & & & 1 \end{bmatrix}$$

- Negativity: $\mathrm{Tr}\,|\rho_A^{R_1}| = \sqrt{2}$
- The SPT invariant

$$\mathrm{Tr}(\rho_A \rho_A^{R_1}) = \frac{1+i}{4} = \frac{1}{2\sqrt{2}} e^{i\pi/4}$$

- This phase is the *many-body* topological invariant of the time-reversal symmetric Kitaev chain.
- It agrees with the partition function of Pin TQFT on $\mathbb{R}P^2$ (Pin^- structure), and can detect the $\mathbb{Z}_8$ classification.