Perspectives in Topological phases:
From Condensed Matter to High-Energy Physics
ICISE Quy Nhon Vietnam, July 15-21, 2018

Anomaly and Symmetry-Protected Critical Phases

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based on

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with

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arXiv:1805.06885 with

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Symmetry&Topology in CMP

"New" classification of phases of matter (quantum many-body systems)

Topologically ordered phases

Symmetry-Enriched Topological (SET) phases

Symmetry-Protected Topological (SPT) phases

incl. Topological Insulators & Topological Superconductors

Classification of **gapped** phases

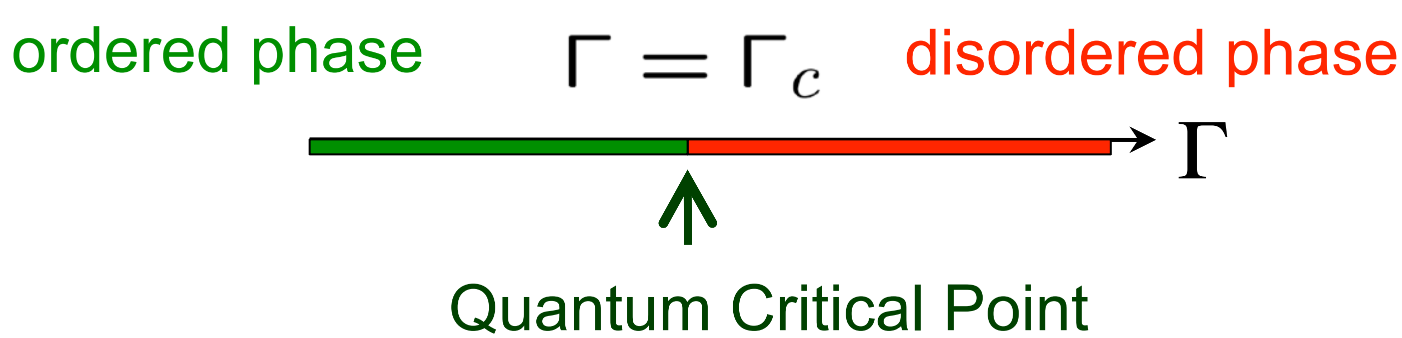
Symmetry&Topology in CMP

“New” classification of gapless, critical phases?

Gapless phases appear at quantum critical points

e.g. (quantum) transverse Ising model

$$\mathcal{H} = -\sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z - \Gamma \sum_j \sigma_j^x$$



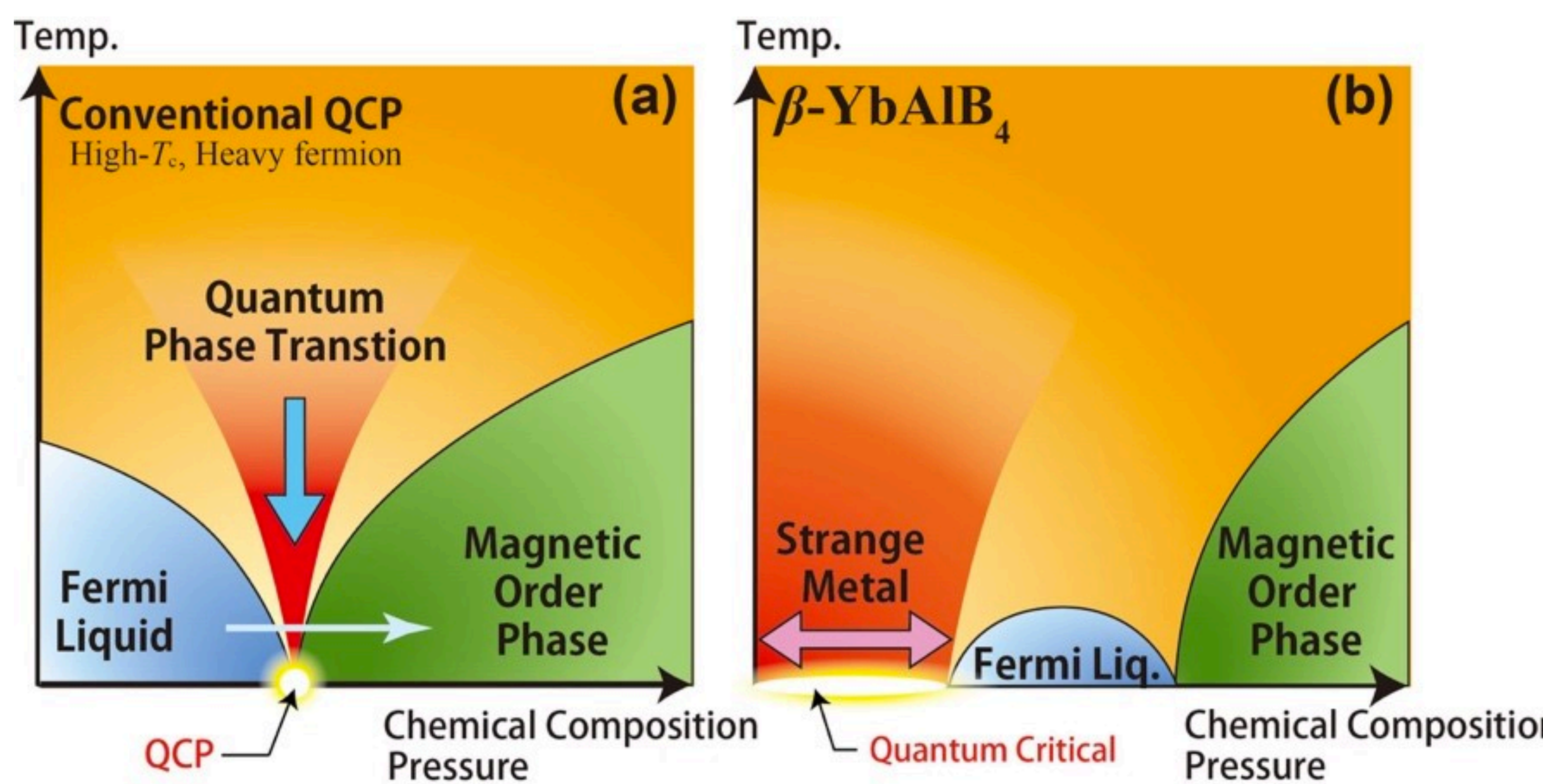
critical point = RG fixed point

relevant perturbation $\rightarrow$ gap

Gapless Critical Phases

But quantum critical phases often appear in cond-mat physics without any apparent fine-tuning

- metallic systems
- Dirac/Weyl semimetals
- β-$YbAlB_4$



[Nakatsuji Group, ISSP]

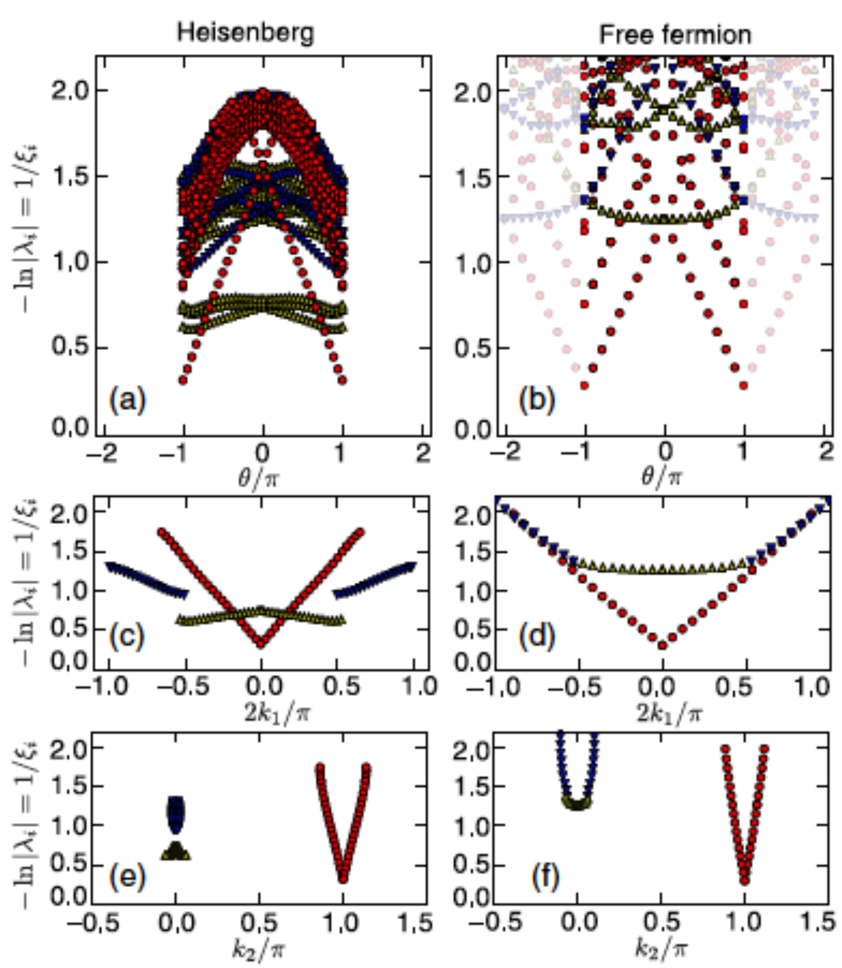
Gapless Critical Phases

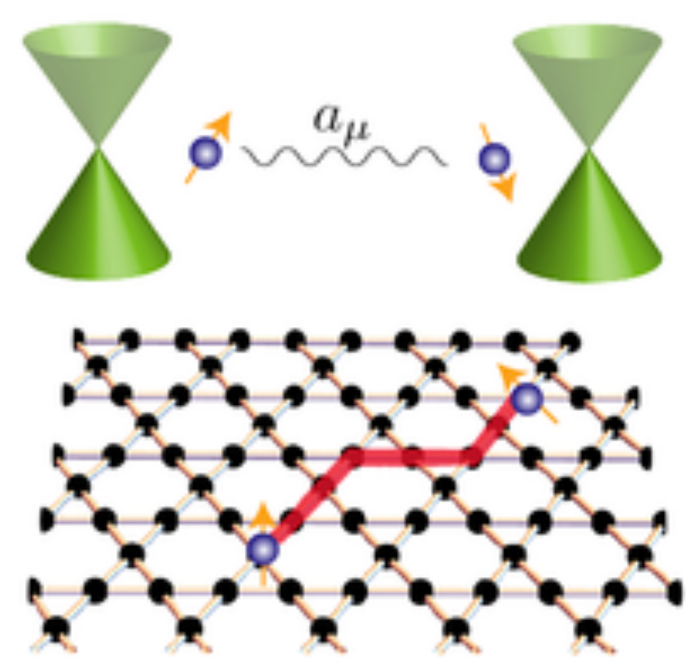
- Kagome spin liquid (*S*=1/2 antiferromagnet):
  Dirac spin liquid?

PHYSICAL REVIEW X **7**, 031020 (2017)

Signatures of Dirac Cones in a DMRG Study of the Kagome Heisenberg Model

Yin-Chen He,[1,2,3] Michael P. Zaletel,[4,3,6] Masaki Oshikawa,[5,3] and Frank Pollmann[1,3,7]





Gapless Critical Phases

- Why are they stable?
- Classification/characterization of these phases

Lieb-Schultz-Mattis Type Theorems

- Many-particle system on a periodic lattice (& periodic b.c.)
- Particle number conserved
- ν particles per unit cell

more general cases:
Po, Watanabe, Cho, Hsieh,….

$$\nu = \frac{p}{q}$$

gapless

~~gapped, with unique ground state~~

gapped, with *q*-fold degenerate ground states

spontaneous symmetry breaking

topological order

LSM 1961
Affleck-Lieb 1986
M.O.-Yamanaka-Affleck 1997
M.O. 2000, Hastings 2004, …

This explains the stability of the gapless phases to some extent.
Recent applications: this also constrains the universality class of the gapless phases

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- gapped, with *q*-fold degenerate ground states
  - spontaneous symmetry breaking
  - topological order

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“LSM” constraint on critical phases

- Many-particle system on a periodic lattice (& periodic b.c.)
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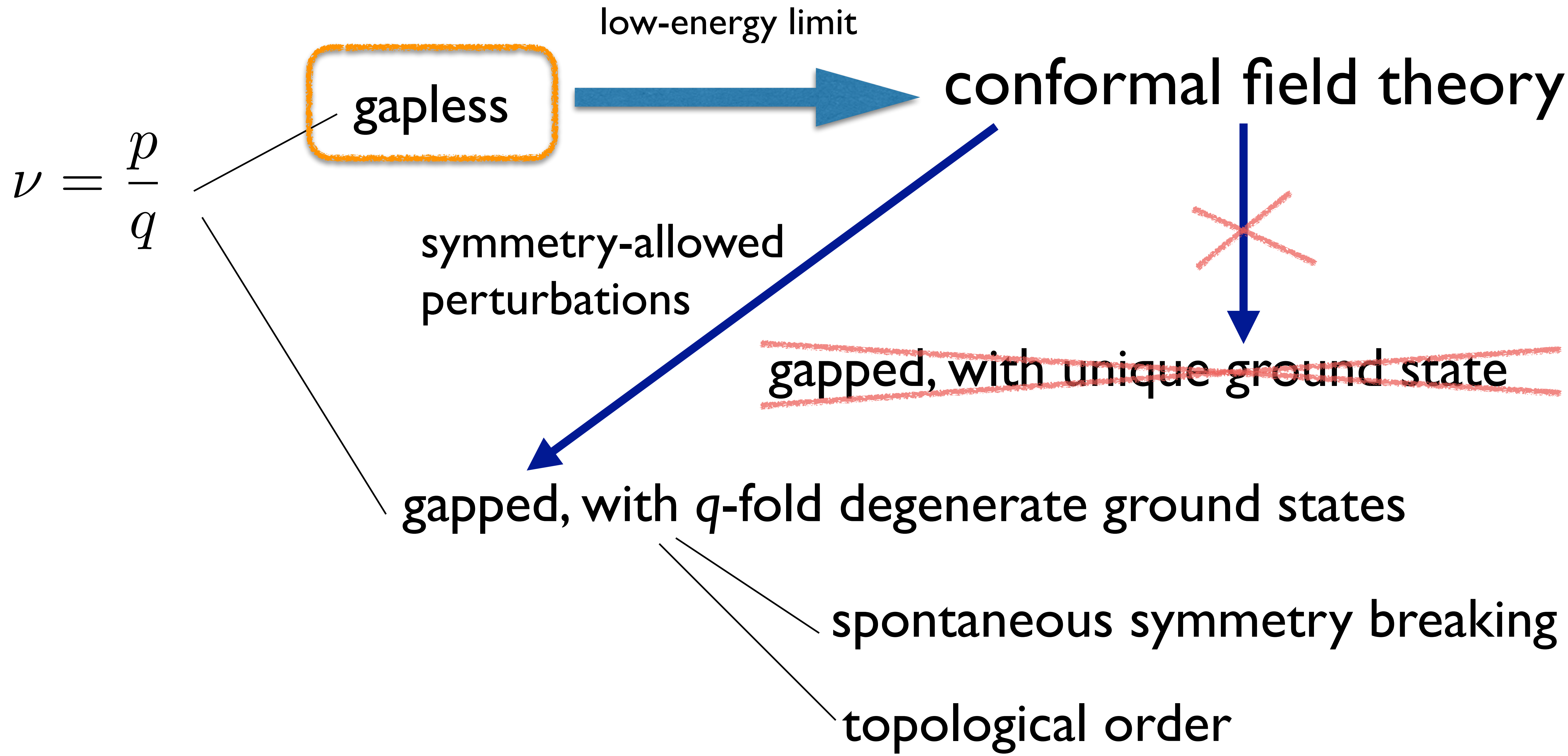

“LSM” constraint on critical phases

For a system subject to a LSM-type constraint,
the conformal field theory describing its gapless phase
cannot be arbitrary

The conformal field theory must inherit the
“ingappability” (LSM-type constraint)
$\Leftrightarrow$ **anomaly matching**

Furuya-M.O. 2017 for SU(2) in 1+1d
Yao-Hsieh-M.O. 2018 for SU(N) in 1+1d

SU(2) AFM chains

Spin-S antiferromagnetic chain with the global SU(2) and lattice translation symmetries

$$\mathcal{H} = \sum_j \left[\vec{S}_j \cdot \vec{S}_{j+1} + J_q \left(\vec{S}_j \cdot \vec{S}_{j+1} \right)^2 + J_2 \vec{S}_j \cdot \vec{S}_{j+2} \cdots \right]$$

Lorentz invariance is expected;

when gapless, low-energy physics should be described by a SU(2) symmetric CFT

$SU(2)_k$ Wess-Zumino-Witten theory

characterized by "level" k = 1, 2, 3, …

Our Claim

In the presence of the SU(2) and
lattice translation (by one site) symmetries,

$$S = 1/2, 3/2, 5/2, \dots$$

- The system is gapped with a SSB of the translation symmetry (doubly degenerate GS)

OR - The system is gapless, described by **$SU(2)_k$ WZW with an odd k**

$$S = 1, 2, 3, \dots$$

- The system is gapped (can be without SSB)

OR - The system is gapless, described by **$SU(2)_k$ WZW with an even k**

Lieb-Schultz-Mattis theorem

In the presence of the U(1) and lattice translation symmetries,

$$S = 1/2, 3/2, 5/2, \dots$$

- The system is gapped with a SSB of the translation symmetry (doubly degenerate GS)

OR - The system is gapless

$$S = 1, 2, 3, \dots$$

- The system is gapped (can be without SSB)

OR - The system is gapless (empty statement!)

Lieb-Schultz-Mattis Theorem

Spin-S antiferromagnetic chain with the global U(1) and lattice translation symmetries

$$\mathcal{H} = \sum_j \left[\vec{S}_j \cdot \vec{S}_{j+1} + J_q \left(\vec{S}_j \cdot \vec{S}_{j+1} \right)^2 + J_2 \vec{S}_j \cdot \vec{S}_{j+2} \cdots \right]$$

if *S* is a half odd integer,

- the system is gapless

  OR

- the ground states are (at least) two-fold degenerate

~~gapped (massive) system with a unique ground state~~

cf.) Haldane gap for integer S

Proof

"Twist operator"

$$U \equiv \exp\left[\frac{2\pi i}{L}\sum_j j S_j^z\right]$$

Apply to the ground state of a finite ring of *L* sites

$$|\Psi_0\rangle$$

$$\langle\Psi_0|U^{-1}\mathcal{H}U|\Psi_0\rangle - \langle\Psi_0|\mathcal{H}|\Psi_0\rangle = O(\frac{1}{L})$$

$|\Psi_1\rangle = U|\Psi_0\rangle$ is a low-energy state!

But this may be just because $|\Psi_1\rangle \sim |\Psi_0\rangle$!

Proof

T : Lattice translation operator

$$TU = UT \exp\left[\frac{2\pi i}{L}\sum_j S_j^z - 2\pi i S_1^z\right] = (-1)^S UT$$

$$(\text{if} \sum_j S_j^z = 0)$$

$$|\Psi_1\rangle = U|\Psi_0\rangle$$

belongs to a different eigenvalue of *T*, and thus

$$\langle\Psi_1|\Psi_0\rangle = 0$$

on-site U(1) and lattice translation symmetries

Haldane Conjecture

Heisenberg Antiferromagnetic Chain

$$\mathcal{H} = J \sum_j \vec{S}_j \cdot \vec{S}_{j+1}$$

$$S = 1/2, 3/2, 5/2, \ldots$$

- The system is gapless (massless)

$$S = 1, 2, 3, \ldots$$

- The system is gapped (massive)

Consistent with the LSM theorem!

(Affleck-Lieb 1986)

Our Claim

In the presence of the SU(2) and
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“Symmetry Protected” gapless phases

SU(2) + Lorentz + lattice translation symmetries

$SU(2)_k$ WZW	**$SU(2)_k$ WZW**
k: even	k: odd

SU(2) WZW theories

Lorentz-invariant critical point:
expect chiral SU(2) x SU(2) symmetry

Natural action with the SU(2) x SU(2) symmetry

$$S_0 = \frac{1}{2\lambda^2}\int d^2x\ \mathrm{Tr}[(g^{-1}\partial_\mu g)^2]$$

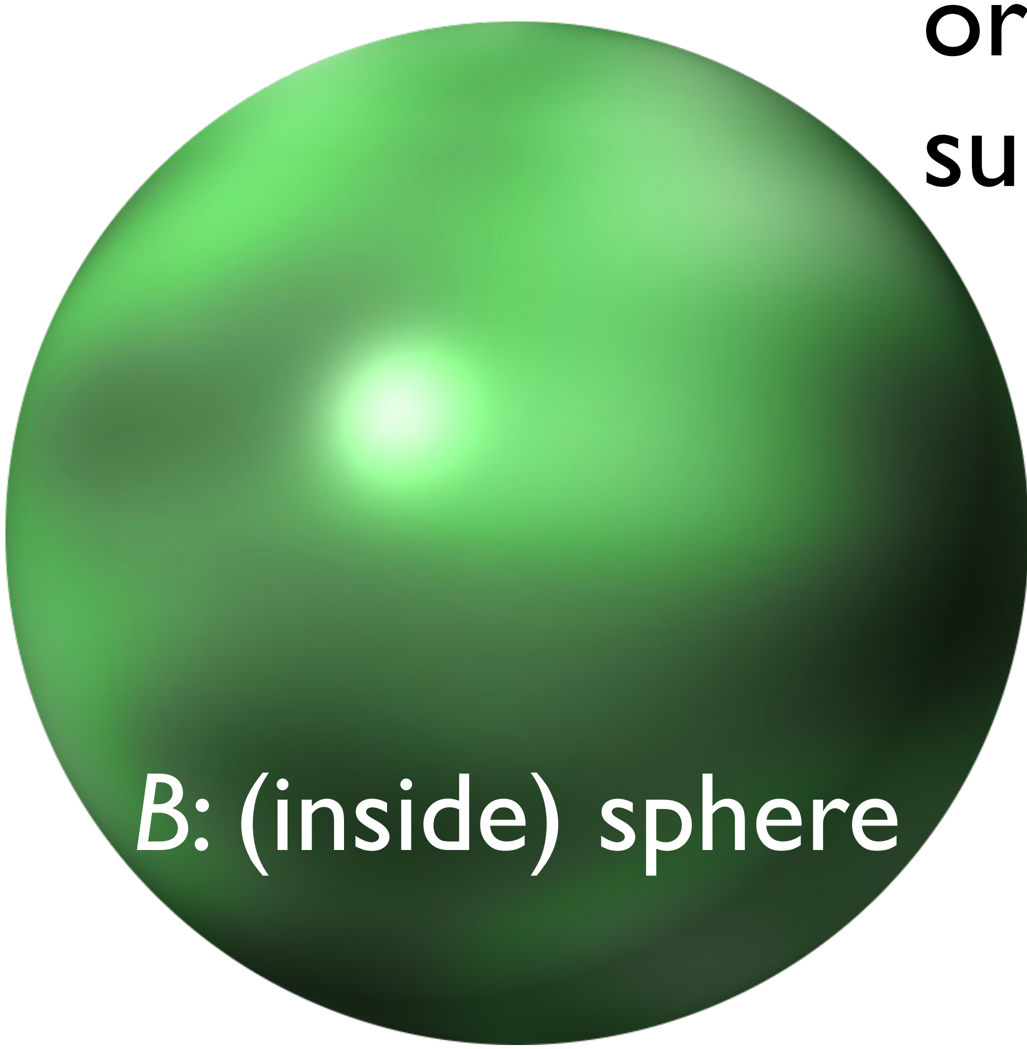
g: SU(2) matrix-valued field

However, RG implies that this theory is
always massive (gapped) "asymptotic freedom"

Wess-Zumino term

$$S = S_0 + k\Gamma_{WZ}$$

$$\Gamma_{WZ} = \frac{1}{12\pi}\int_B d^3x\, \epsilon^{ijk}\mathrm{Tr}[(g^{-1}\partial_i g)(g^{-1}\partial_j g)(g^{-1}\partial_k g)]$$



original space-time:
surface of the sphere

uniqueness of $k\Gamma_{WZ}$
(modulo 2π)
$\Rightarrow$ k: integer

RG has a nontrivial fixed point
if $k \neq 0$ → gapless critical phase

Kac-Moody algebra

$$J(z) = \frac{k}{2} g^{-1} \partial_z g = \sum_n \frac{1}{z^{n+1}} J_n^a \frac{\sigma^a}{2}$$

$$[J_n^a, J_m^b] = i f_{abc} J_{n+m}^c + \frac{1}{2} k n \delta_{ab} \delta_{n+m,0}$$

This "includes" Virasoro algebra (conformal invariance) and is very powerful — determines scaling dimensions (critical exponents) etc.

$$c = \frac{3k}{k+2} \qquad h_j = \frac{j(j+1)}{k+2} \qquad 0 \le j \le \frac{k}{2}$$

central charge

scaling dimension of spin-*j* field

Spin chain and WZW

$$\vec{S}_i \sim \vec{J}_i + \mathrm{const.}(-1)^i \mathrm{tr}(g\vec{\sigma})$$

Lattice translation symmetry

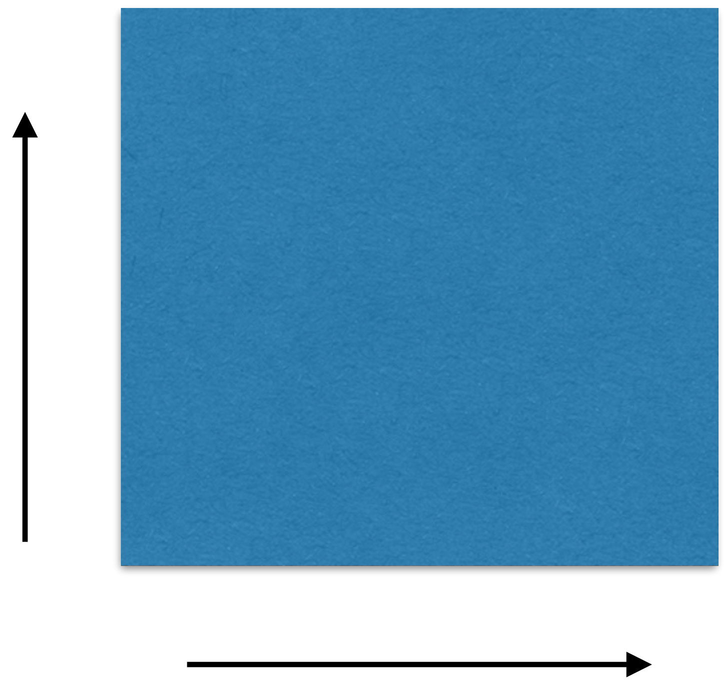
⇔ discrete Z_2 symmetry $g \to -g$

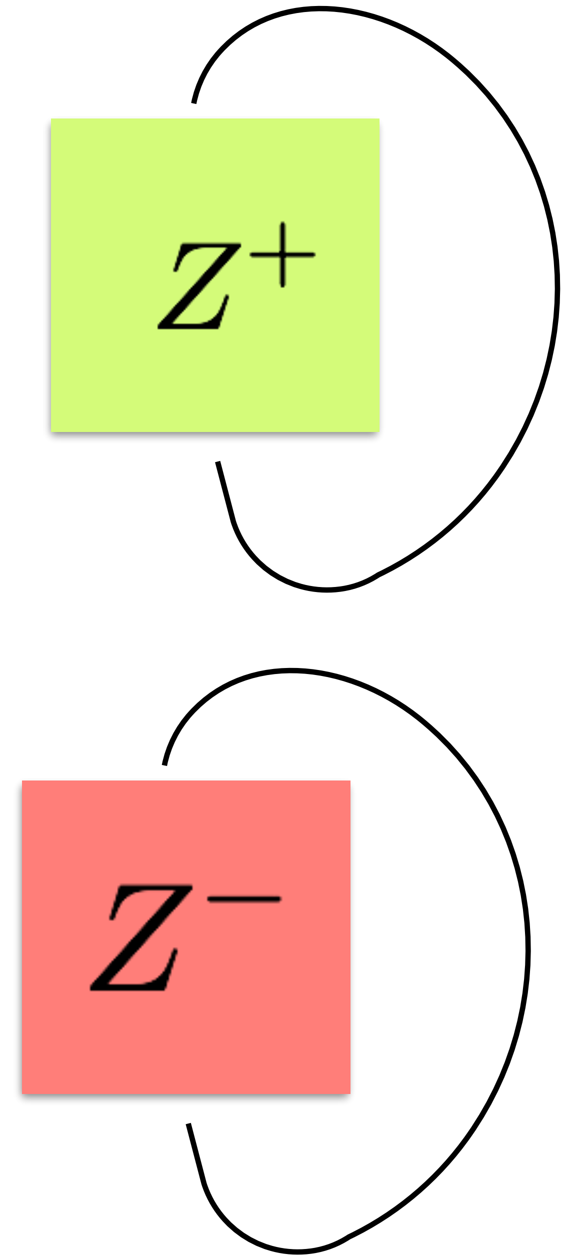
If there is the Z_2 symmetry,

we should be able to consider a projection

to Z_2-symmetric subspace?

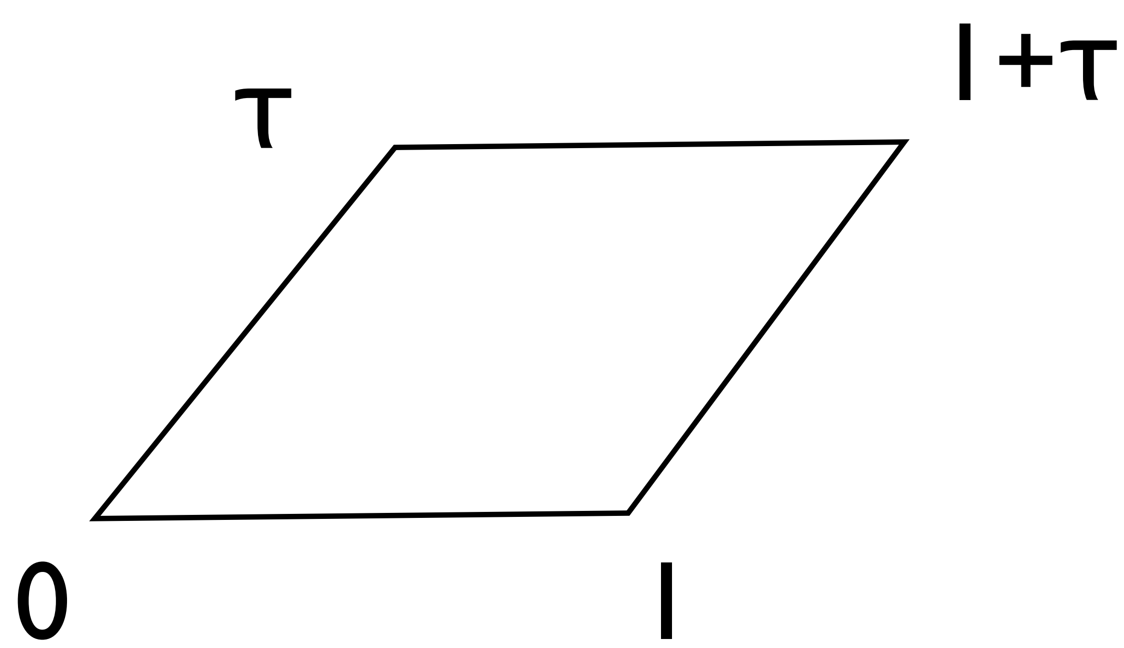
Projection vs. Path Integral





$$Z_+^{\mathrm{proj}} = \mathrm{Tr}[P_+ e^{-\mathcal{H}}] = \frac{1}{2}[Z^+ + Z^-]$$

Modular Invariance



Partition function of a consistent CFT must be invariant under modular transformations generated by

$$\mathcal{S} : \tau \to -1/\tau$$
$$\mathcal{T} : \tau \to \tau + 1$$

Orbifold Construction

The "projected" partition function Z_+^{proj} is not modular invariant by itself — must be supplemented by twisted sectors

$$Z_+ = (1 + \mathcal{S} + \mathcal{T}\mathcal{S}) Z_+^{\mathrm{proj}} - Z_{\mathrm{WZW}}$$

The resulting partition function represents the "Z_2 orbifold" of the original $SU(2)_k$ WZW theory

cf.) S. Ryu et al.
protection of edge states ↔ gapped SPT phase

Global Anomaly

The Z_2 orbifold should be modular invariant by construction — but this is NOT always the case!

The Z_2 orbifold is **modular invariant if *k* is even**, but it is **modular NON-invariant if *k* is odd**

Gepner-Witten 1986

STRING THEORY ON GROUP MANIFOLDS

Doron GEPNER and Edward WITTEN

Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08544, USA

Received 26 May 1986

What does this mean?

If the orbifold is modular invariant, we can consider projection to the symmetric sector, and open a gap within that sector

However, if it is modular non-invariant, we cannot open the gap to obtain a unique ground state within the symmetric sector

Global anomaly =
"ingappability" in the presence of the symmetry
(S. Ryu et al. on edge theory)

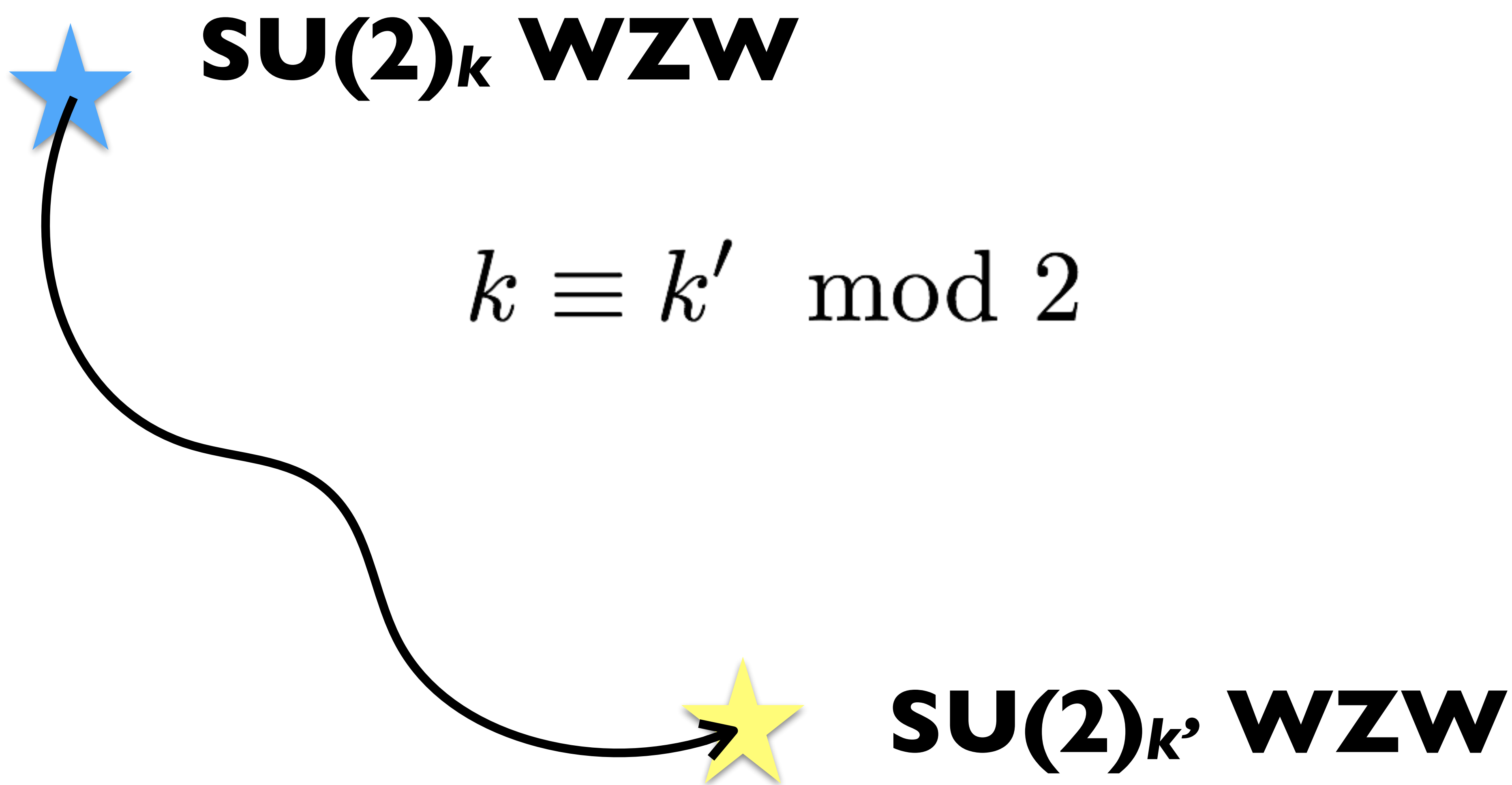
Selection Rule

Perturb $SU(2)_k$ WZW with SU(2) and Z_2-symmetric relevant operators; suppose the RG flow reaches $SU(2)_{k'}$ WZW

if k is even, we should be able to consider the projection to Z_2 symmetric sector; the RG flow can be understood in terms of the Z_2 orbifold → k' is also even

if k is odd, the IR fixed point should also have the global anomaly (otherwise contradicts with LSM) → k' is also odd

In terms of RG…



SU(2)$_0$ WZW is identified with gapped phase with a unique ground state

Spin Chains and WZW

There is a special integrable (Bethe-ansatz solvable) spin chain model for any *S*
(Takhtajan-Babujian model)

e.g. for *S*=1: $$\mathcal{H}_{TB} = \sum_j \left[\vec{S}_j \cdot \vec{S}_j - (\vec{S}_j \cdot \vec{S}_j)^2 \right]$$

It is known that this can be described by $SU(2)_{2S}$ WZW

Other models can be regarded as Takhtajan-Babujan + perturbations

Examples

In fact, all the spin chain examples (known to us) are consistent with our "selection rule"

$$\mathcal{H}_{J_1\text{-}J_3} = \sum [J_1 \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1} + J_3 \{ (\boldsymbol{S}_{j-1} \cdot \boldsymbol{S}_j)(\boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}) + \text{H.c.} \}]$$

realizes $SU(2)_{2S}$ even though this is not integrable

Michaud et al. 2012

However, it is important to emphasize that our selection rule is valid only in the presence of the lattice translation (by one site) symmetry

Our Claim

In the presence of the SU(2) and lattice translation symmetries,

$$S = 1/2, 3/2, 5/2, \ldots$$

- The system is gapped with a SSB of the translation symmetry (doubly degenerate GS)

OR - The system is gapless, described by

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Lieb-Schultz-Mattis Theorem

You CAN open a gap for odd k, but only if you include both symmetric and antisymmetric sectors

→ degenerate ground states

= spontaneous breaking of the Z_2 symmetry

"Lieb-Schultz-Mattis theorem" for $SU(2)_k$ WZW with an odd k

Breaking the Z_2 symmetry

$$\mathcal{H}_{J_1\text{-}J_3\text{-}\delta} = \mathcal{H}_{J_1\text{-}J_3} - J_1\delta\sum_j(-1)^j \boldsymbol{S}_j\cdot\boldsymbol{S}_{j+1}$$

$\delta \neq 0$ breaks the lattice translation symmetry explicitly (one site→two sites)

This is equivalent to breaking of the discrete Z_2 symmetry in the WZW theory

For S=1, a RG flow from $SU(2)_2$ WZW to $SU(2)_1$ WZW was indeed confirmed numerically

Kitazawa-Nomura 1999

SU(N) chains

$$\mathscr{H}_{\mathrm{HAF}} = J \sum_{\langle i,j\rangle,\alpha,\beta} S^{\alpha}_{i,\beta} S^{\beta}_{j,\alpha}, + \ldots.$$

$$\left[S^{\alpha}_{i,\beta}, S^{\gamma}_{j,\delta}\right] = \delta_{i,j}\left(\delta^{\alpha}_{\delta} S^{\gamma}_{i,\beta} - \delta^{\gamma}_{\beta} S^{\alpha}_{i,\delta}\right)$$

Same strategy as in SU(2):
orbifolding of the Z_N symmetry?

not powerful enough…. (can't reproduce LSM)
more systematic approach?

Symmetry of the "SU(N)" chain

… is not exactly SU(N)

$$e^{2\pi i m/N}\mathbb{1} \in \mathrm{SU}(N) \quad m = 0, 1, \ldots, N-1$$

"center" of SU(N) = Z_N do not change the quantum state

On-site symmetry of SU(N) chain:

PSU(N) = SU(N)/Z_N

Symmetry of the system:

PSU(N) × Z ← lattice translation

Gauging the Symmetry

Orbifolding = gauging the symmetry

't Hooft anomaly:

anomaly obstruction in gauging a global symmetry

NO anomaly for "on-site" symmetry of a lattice model, since the corresponding gauge field can be introduced at the lattice level in a perfectly well-defined way

However, the translation symmetry is NOT an "on-site" symmetry, and could show a 't Hooft anomaly

$\Leftrightarrow$ LSM constraint

Cho-Hsieh-Ryu 2017

Classification of ’t Hooft anomaly

’t Hooft anomaly for PSU(*N*)×Z in 1+1 D

⇔ “ingappable” phase in 1+1D protected by PSU(*N*)×Z

⇔ SPT phase with PSU(*N*)×Z symmetry in 2+1 D

$$H^3(\mathrm{PSU}(N)\times\mathbb{Z}, U(1)) \cong H^3(\mathrm{PSU}(N), U(1)) \oplus H^2(\mathrm{PSU}(N), U(1))$$

SPT phase protected by PSU(N) only (anomalous edge state not realizable in 1+1D)

SPT phase protected by PSU(N) & Z

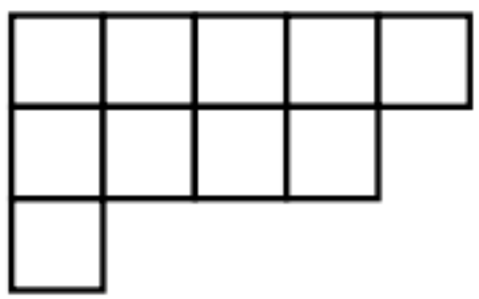
⇔ **mixed anomaly**

⇔ LSM constraint

LSM index for SU(N) chain

mixed anomaly factor $H^2(\mathrm{PSU}(N), U(1)) = \mathbb{Z}_N$

general SU(N) spin representation at each site:



Young tableau

$\mathcal{I}_N =$ (total number of boxes in the Young tableau) mod *N*

$$\in H^2(\mathrm{PSU}(N), U(1))$$

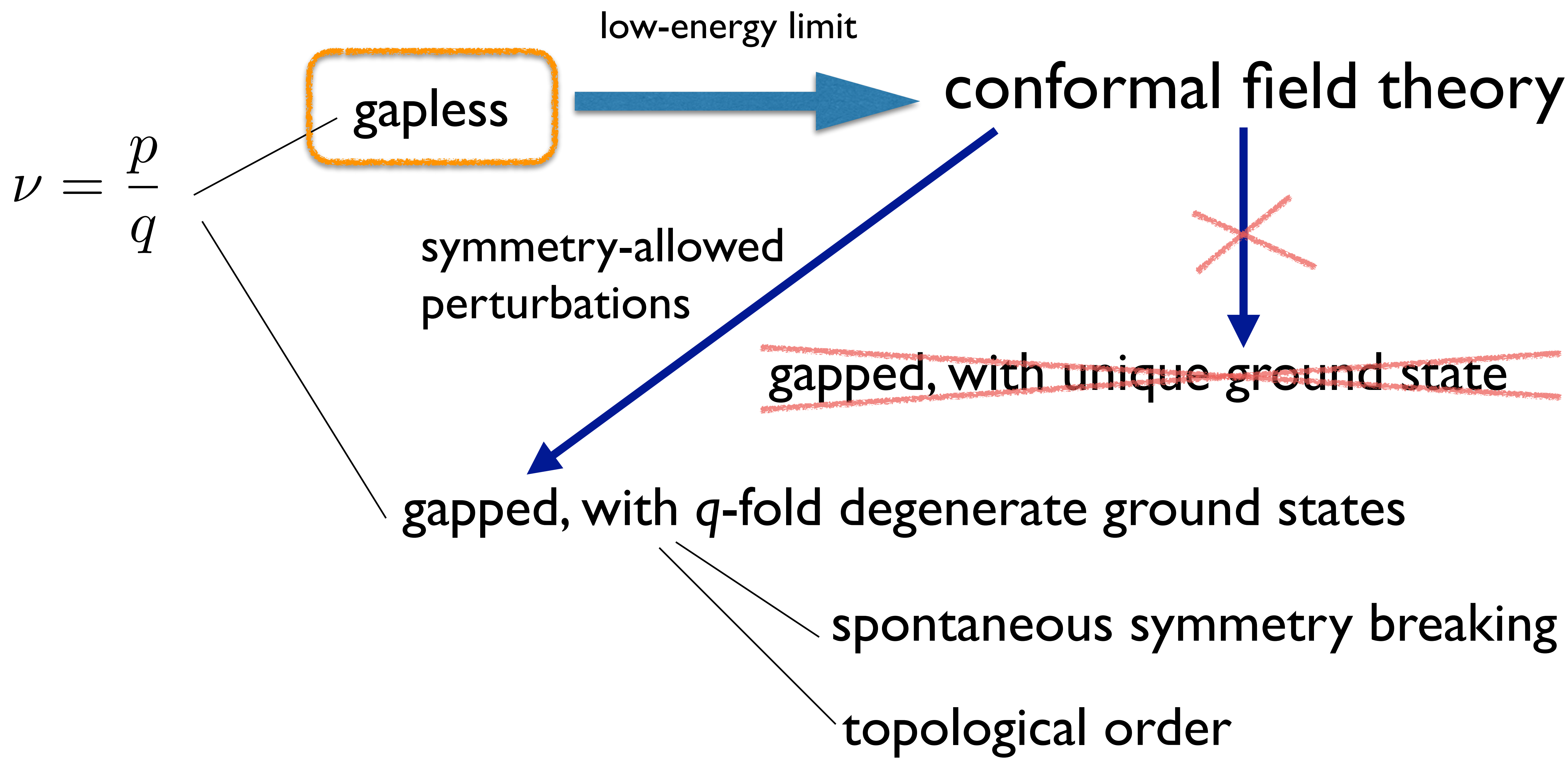
$\mathcal{I}_N \neq 0$ ⇒ LSM constraint

(gapless or $\mathcal{I}_N$ -fold degenerate g.s.)

reproduces Affleck-Lieb 1986

“LSM” constraint on critical phases

- Many-particle system on a periodic lattice (& periodic b.c.)
- Particle number conserved
- ν particles per unit cell



CFT for Gapless Critical Phases

SU(*N*) Wess-Zumino-Witten CFT, level *k*

$$g(x,t) \in \mathrm{SU}(N)$$

Lattice translation symmetry

$$\Leftrightarrow \qquad g \to e^{2\pi i m/N} g$$

Mixed PSU(*N*)-Z_N anomaly: $km \mod N$

Anomaly matching with the lattice model

$$km \equiv \mathcal{I}_N \mod N$$

$$(km \equiv \text{total number of boxes in YT} \mod N)$$

Symmetry-Protected Critical Phases

with PSU(N) × translation symmetry:

N distinct symmetry-protected classes characterized by $km \mod N$

N=2, m=1 ⇒ recover the previous result on SU(2)

Application to Enlarged Symmetry

Systematic description of LSM constraint in terms of mixed anomaly: handling of enlarged symmetry

Lattice model: PSU(N) × translation, with $\mathcal{I}_N$

SU(*N*')$_{k'}$ WZW with lattice translation $g \to e^{2\pi i m'/N'} g$

Consider *N*'/gcd(*N*', *k*'*m*') copies ⇒ anomaly free

Anomaly matching requires
the lattice model is also anomaly-free

$$\mathcal{I}_N \frac{N'}{\gcd(N', k'm')} \equiv 0 \mod N$$

Application to Enlarged Symmetry

$$\mathcal{I}_N \frac{N'}{\gcd(N', k'm')} \equiv 0 \mod N$$

Example:

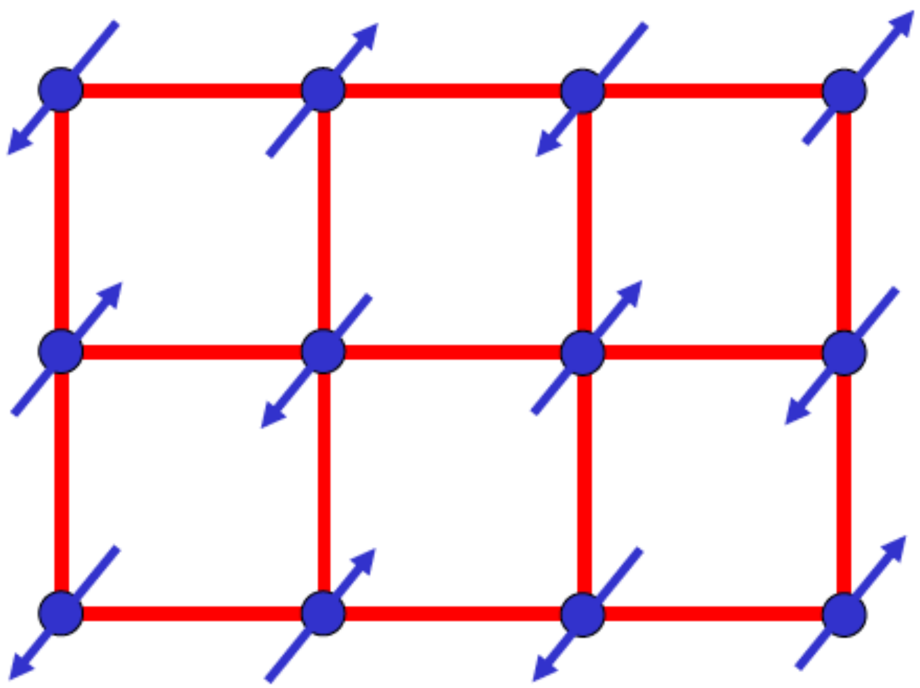
N=2, *N'*=3 (PSU(2) → PSU(3) enlargement)

$\mathcal{I}_N$ must be even (integer spin per unit cell!)

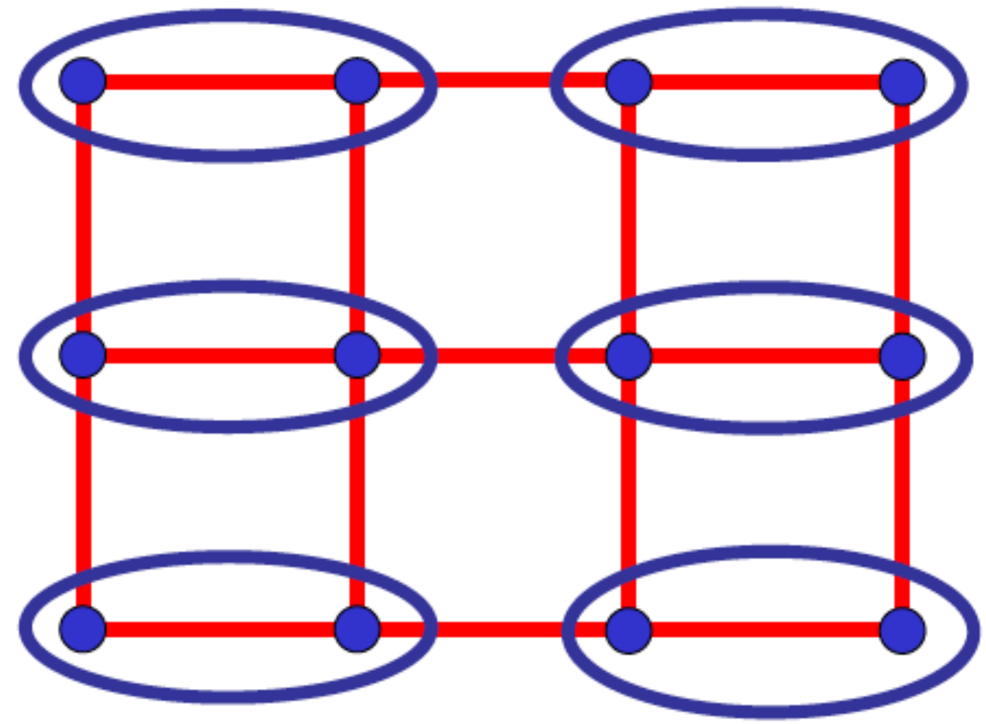
Higher Dimensions: DQCP

Senthil-Vishwanath-Balents-Sachdev-Fisher 2003〜

S=1/2 Square Lattice Antiferromagnet

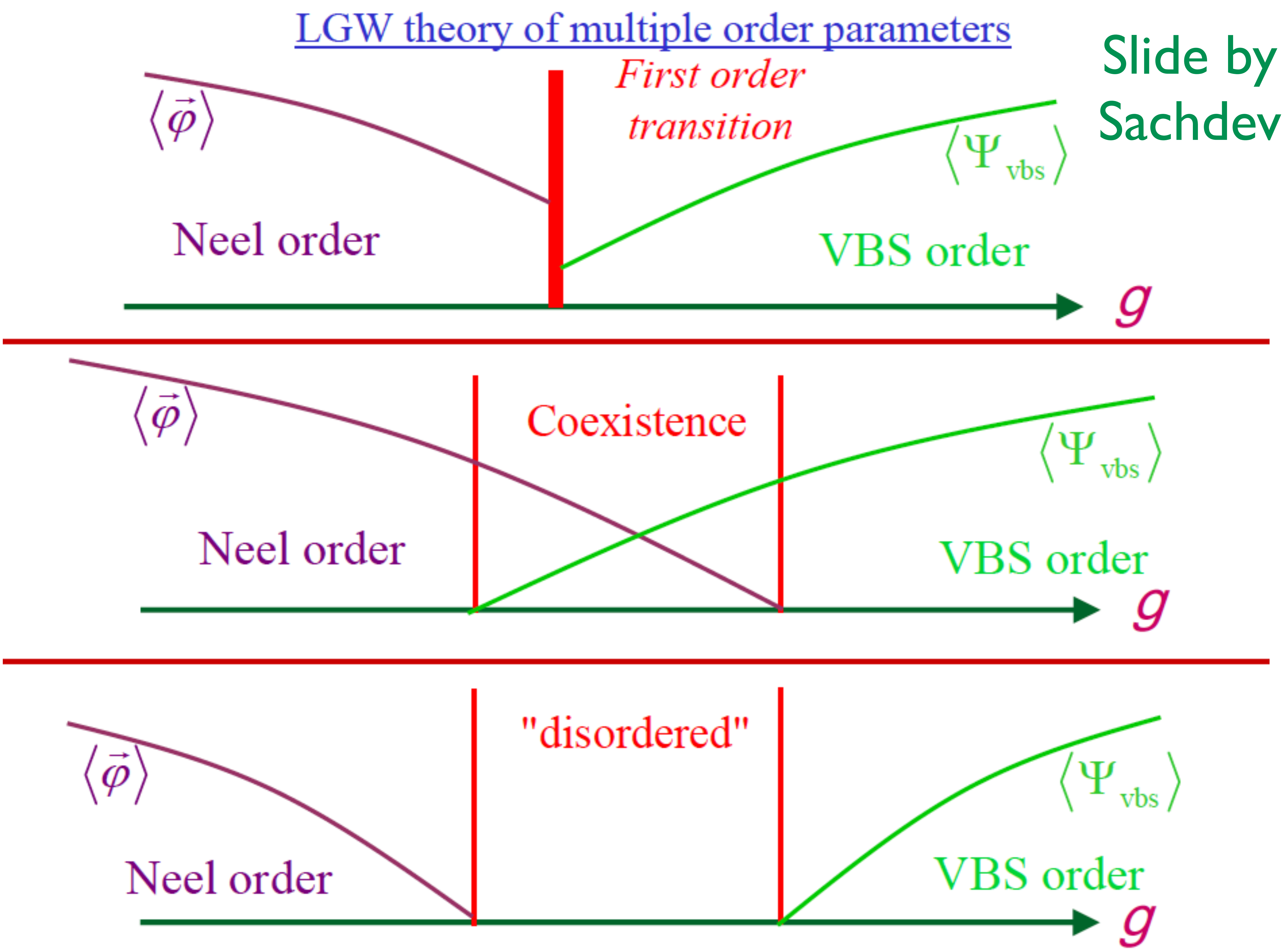


Néel phase



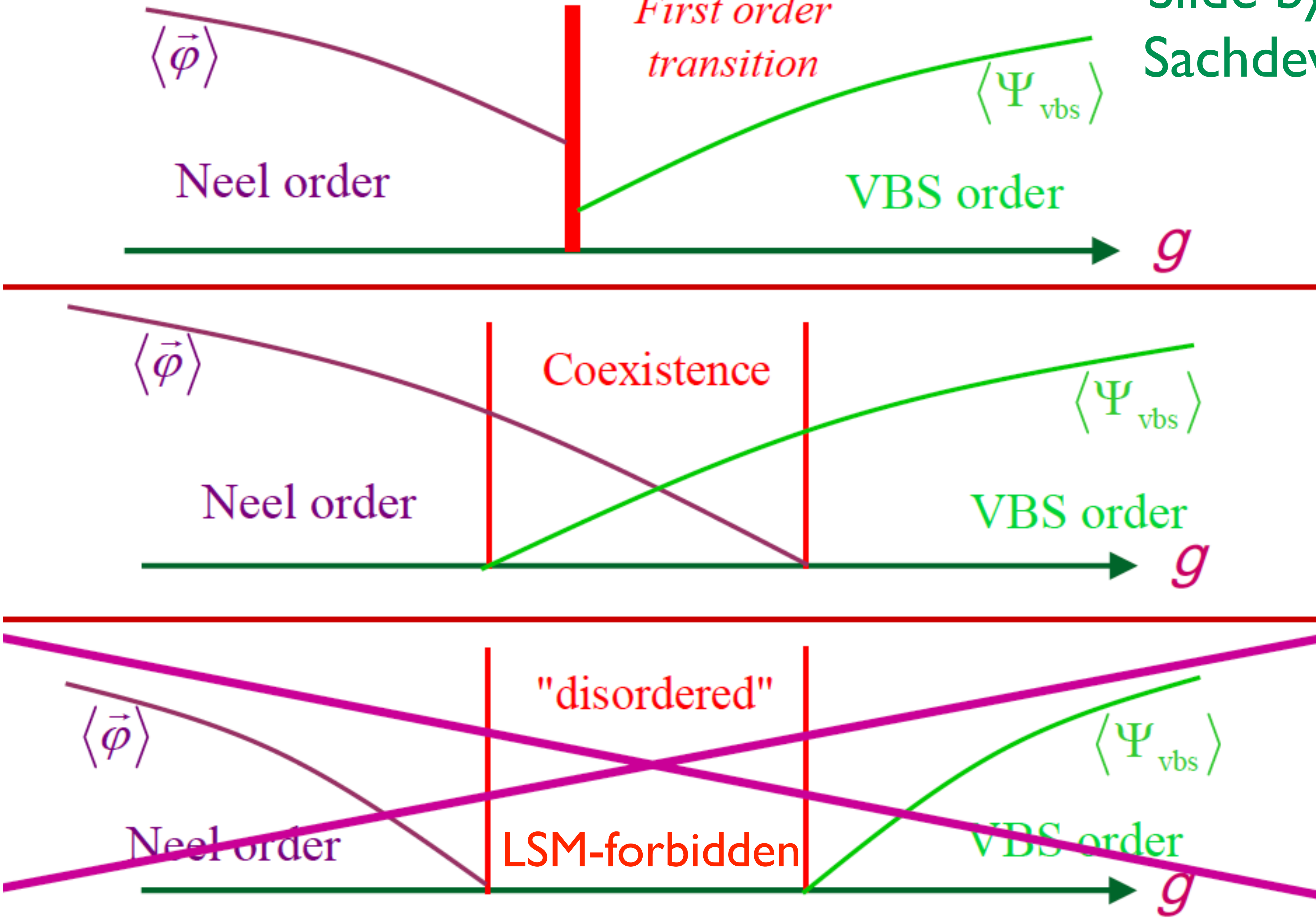
Valence-Bond Crystal phase

Quantum Critical Point between the two?


LGW theory of multiple order parameters
Slide by Sachdev
$\langle\vec{\varphi}\rangle$
First order transition
$\langle\Psi_{vbs}\rangle$
Neel order
VBS order
g
$\langle\vec{\varphi}\rangle$
Coexistence
$\langle\Psi_{vbs}\rangle$
Neel order
VBS order
g
"disordered"
$\langle\vec{\varphi}\rangle$
$\langle\Psi_{vbs}\rangle$
Neel order
VBS order
g

LGW theory of multiple order parameters

Slide by Sachdev



Anomaly at DQCP

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Deconfined Quantum Critical Points: Symmetries and Dualities

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Intrinsic and emergent anomalies at deconfined critical points

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(Dated: July 26, 2017)

Summary

- Symmetry & Topology has interesting implications not only on gapped phases but also on gapless critical phases
- When LSM-type theorem is applicable, it does not only protect gapless nature, but also constrains possible universality classes (“anomaly matching”)
- LSM-type constraint corresponds to “mixed ’t Hooft anomaly” between the on-site and translation symmetries
- N distinct symmetry-protected classes of gapless critical phases corresponding to $SU(N)_k$ in 1+1D
- Many interesting questions to be explored