Generalized U(1) gauge theories and subdimensional dynamics

Maissam Barkeshli
UMD / JQI

Perspectives in Topological phases: From Condensed Matter to High-Energy Physics
Quy Nhon, Vietnam
7/19/2018

Daniel Bulmash, MB, Phys. Rev. B 97, 235112 (2018)
D. Bulmash, MB, **1806.01855**

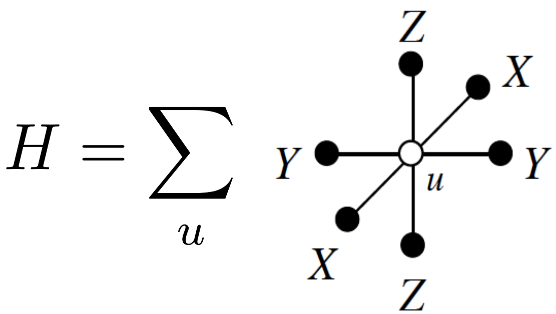
Recently a wide class of theoretical models have been discovered exhibiting quasiparticles with highly constrained dynamics

- Gapped phases: fracton models

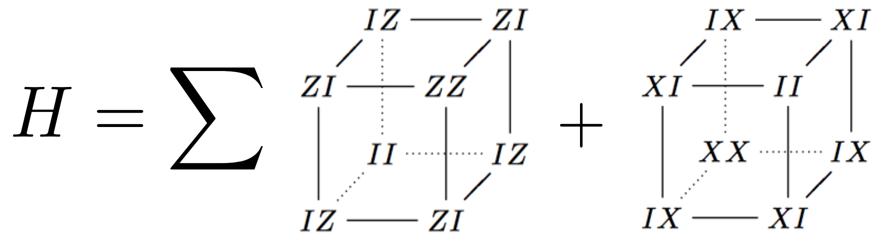
- Gapless phases: higher rank symmetric U(1) gauge theories

Challenges our understanding of phases of matter, universality, and field theory

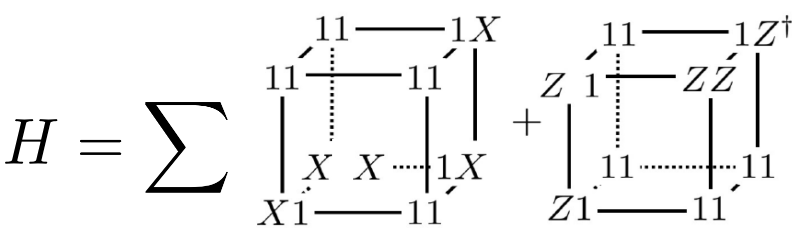
Gapped phases: fracton models



Chamon model. Chamon (2005)
Bravyi, Leemhuis, Terhal (2010)



Cubic Code, Haah (2011)
Bravyi-Haah (2011)



Sierpinski prism model.
Yoshida (2013)

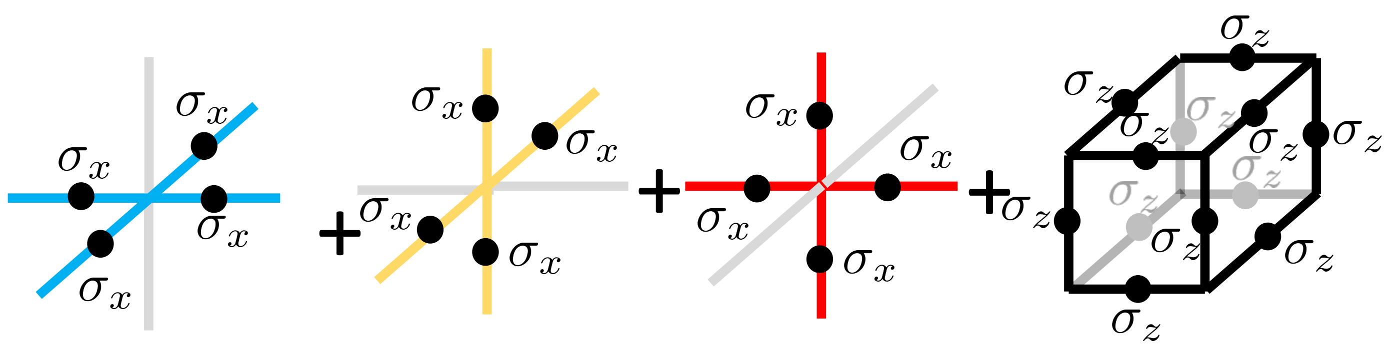
General classification of (gapped) commuting Pauli Hamiltonians

Haah 2012, Yoshida 2013

Vijay, Haah, Fu PRB **94** 235157 (2016)

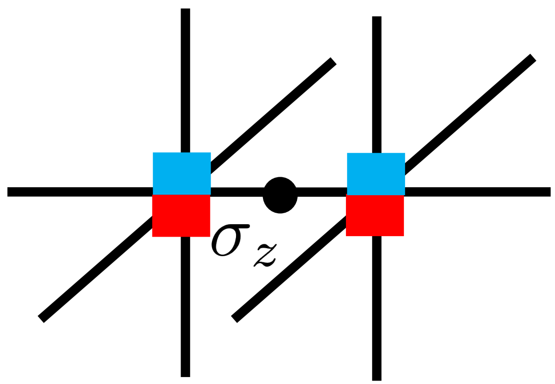
X-Cube Model

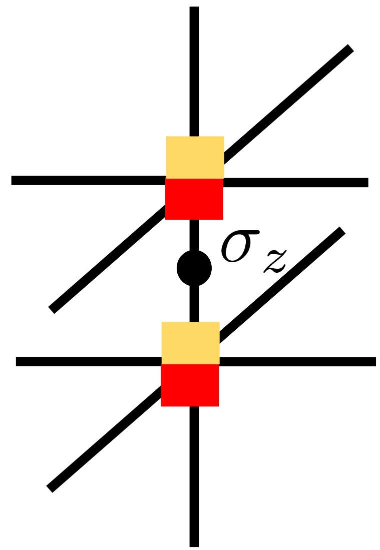
Spin-1/2 on links of cubic lattice

$H = -\sum$ 

- Gapped
- Ground states have +1 eigenvalue for every term in Hamiltonian
- "Topological" degeneracy 2^{6L-3}
- Contains 0D, 1D, and 2D excitations

One-Dimensional Excitations of X-Cube

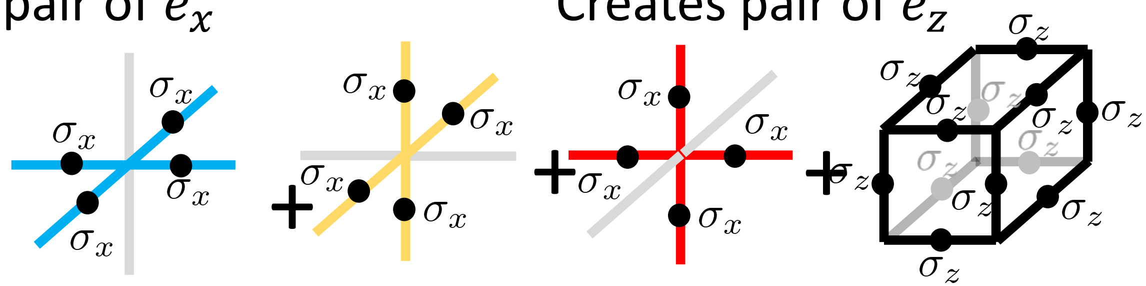




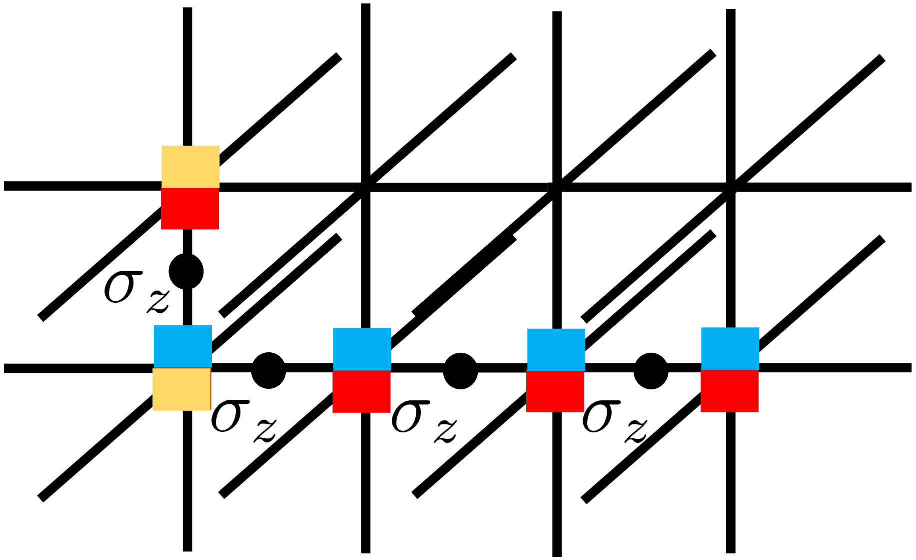
Creates pair of e_x

Creates pair of e_z

$$H = -\sum$$

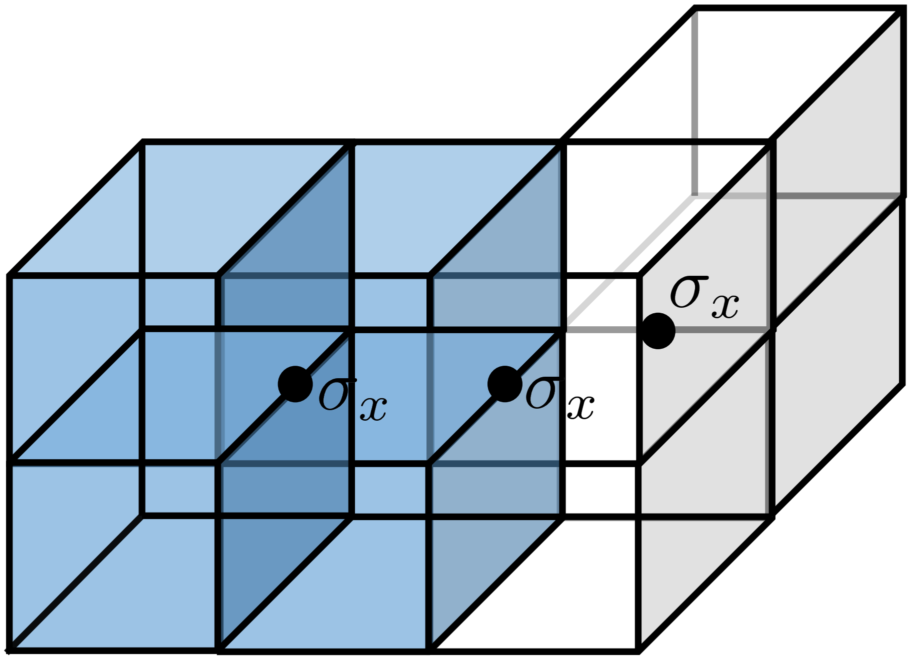


One-Dimensional Excitations of X-Cube

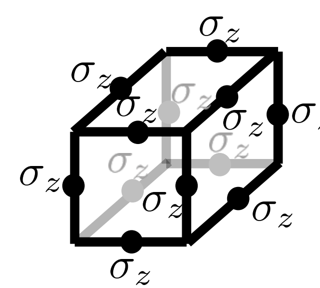


e_x only mobile in x direction!

2D Excitations of X-Cube Model



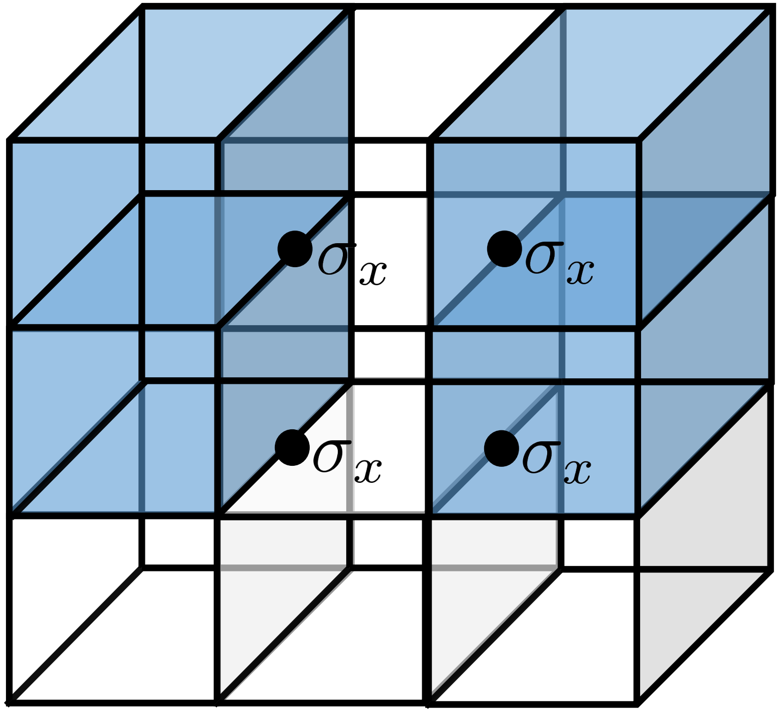
$H = -\sum$



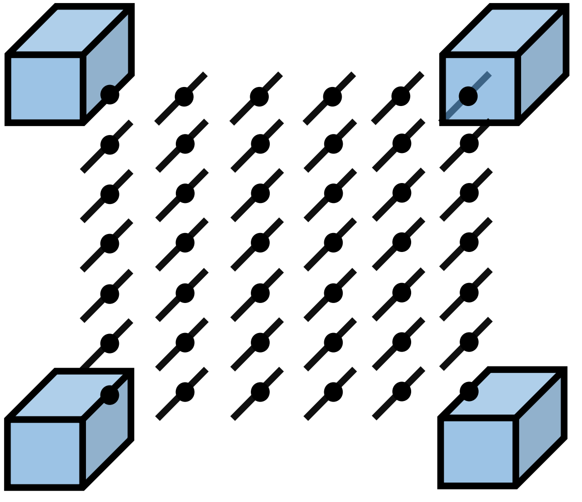
$+ \cdots$

m_{xy} mobile in xy-plane

Immobile (0D) Excitations of X-Cube Model



Immobile (0D) Excitations of X-Cube Model



Isolated excitations live at the corners of membrane operators and are immobile.

Many interesting questions about gapped fracton phases:

1. What is the general theoretical framework that properly captures all the possible phases and their properties? (c.f. Gauge theory, Unitary modular tensor category theory)

2. Can it occur in 2+1 D , or only in higher dimensions? What are the restrictions on dimensionality in space?

3. To what extent can “field theory” describe the universal properties of these phases?

4. Nature of phase transitions from more conventional phases?

5. Possible physical realizations?

Gapless systems: higher rank symmetric U(1) gauge theories

Xu et. al, 2006,2008, 2010, 2013
Rasmussen et. al. 2016
Pretko 2016; Bulmash, MB, 2018

Example: Rank 2, Scalar charge theory:

"Electric field" $E_{ij} = E_{ji}$ "Gauge field" $A_{ij} = A_{ji}$

Gauss law $\sum_{i,j} \partial_i \partial_j E_{ij} = \rho$ $A_{ij} \to A_{ij} - \partial_i \partial_j \alpha$

Non-symmetric
Rank-2 magnetic field

$$B_{ij} = \begin{cases} \sum_{ab} \epsilon_{iab} \partial_a A_{bi} & i = j \\ \sum_{a \neq i,j} (\partial_j A_{aj} - \partial_a A_{jj}) & i \neq j \end{cases}$$

$$\mathcal{H} \sim E^2 + B^2 \qquad \omega \sim k \qquad \mathcal{L} \sim (\partial_i \partial_j A_0 - \partial_t A_{ij})^2 - B^2$$

Pure gauge theory is non-relativistic

Constrained dynamics of charges:

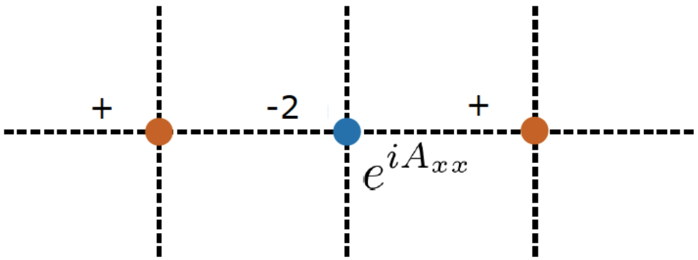
Pretko 2016

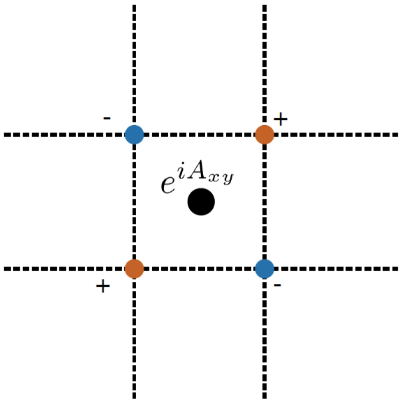
Conservation of total dipole moment:

$$\int d^3x\, x_k \rho = \int d^3x\, x_k \partial_i \partial_j E_{ij} = 0$$

Motion of single charges violates conservation of dipole moment

Isolated charges are immobile. Dipoles are mobile in all directions





Isolated charges are also strongly confined energetically

$$\partial_i \partial_j E_{ij} = \rho \Rightarrow E \sim 1/k^2$$

$$H \sim \int E^2 \sim \int d^3k \, 1/k^4 \sim R$$

Energy cost to isolate charges by a distance R is linear in R

Confinement due to electrostatics, not instanton effect

Field theory + matter

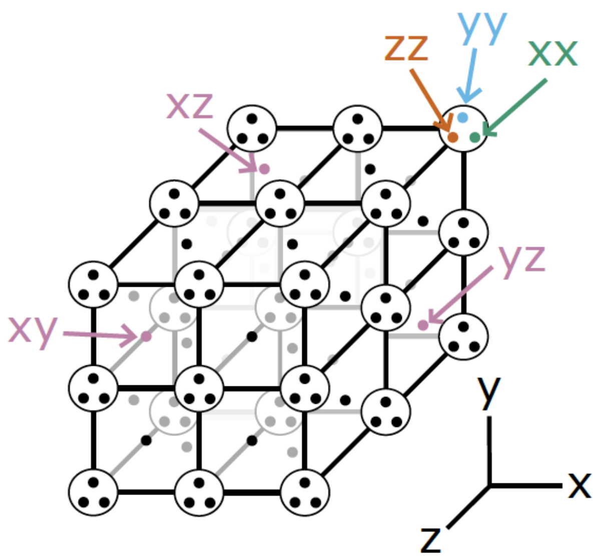
$$\mathcal{H} = E^2 + B^2 + \sum_{ij} U \cos(\partial_i \partial_j \phi + A_{ij})$$

The matter coupling is intrinsically interacting.
(and breaks continuous rotational invariance)

Therefore the theory cannot be defined perturbatively, or as a relevant deformation of a UV fixed point

Only known definition is based on a lattice

Lattice definition: Local rotor model



(b)

$$H = H_{Maxwell} + H_{Higgs} + H_{Gauss}$$

$$H_{Maxwell} = \sum_{\mathbf{r},i} \left(\tilde{h}_s E_{ii}^2 - \frac{1}{g_s^2} \cos(B_{ii}) \right) + \sum_{\mathbf{r},i<j} \left(\tilde{h}_f E_{ij}^2 - \frac{1}{g_f^2} \cos(B_{ij}) \right)$$

$$H_{Higgs} = \sum_{\mathbf{r}} \frac{L(\mathbf{r})^2}{2M} - V \sum_{i<j} \cos(\Delta_i \Delta_j \theta + p A_{ij})$$

$$H_{Gauss} = \tilde{U} \sum_{a,\mathbf{r}} (G(E) - pL_a)^2$$

$$G(E) = \sum_{ij} \Delta_i \Delta_j E_{ij}$$

E_{ii} defined on sites

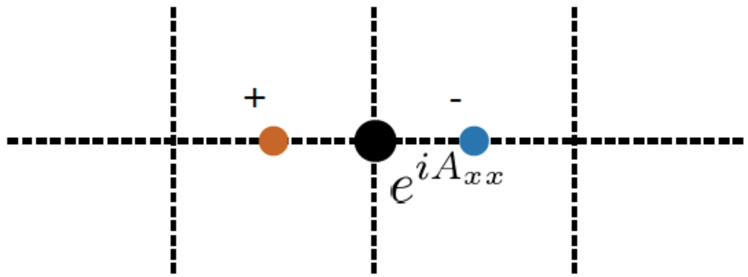
E_{ij} for $i \neq j$
defined on faces

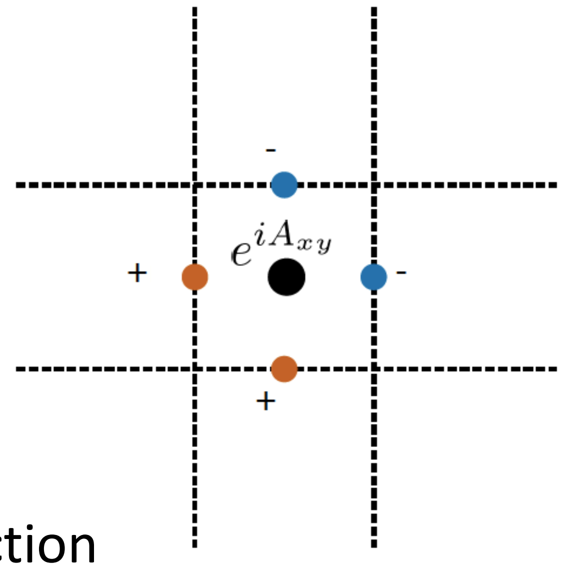
Gauss law can be enforced as an energetic constraint

Vector charge theories:

$$2\sum_i \Delta_i E_{ij} = \rho_j$$

$$A_{ij} \rightarrow A_{ij} + \Delta_i \alpha_j + \Delta_j \alpha_i$$





i oriented charges can only move in i direction

$$B_{ij} = \begin{cases} \frac{1}{2}\sum_{a\neq b\neq i}\left(2\Delta_a^2 A_{bb} - \Delta_a\Delta_b A_{ab}\right) & i = j \\ \frac{1}{2}\sum_{k\neq i,j}\left[\left(\Delta_i\Delta_k A_{jk} + \Delta_j\Delta_k A_{ik} - \Delta_k^2 A_{ij}\right) - 2\Delta_i\Delta_j A_{kk}\right] & i \neq j \end{cases}$$

$$\mathcal{H} \sim E^2 + B^2 \qquad \omega \sim k^2$$

What is the relation between the higher rank symmetric gapless gauge theories and gapped fracton models?

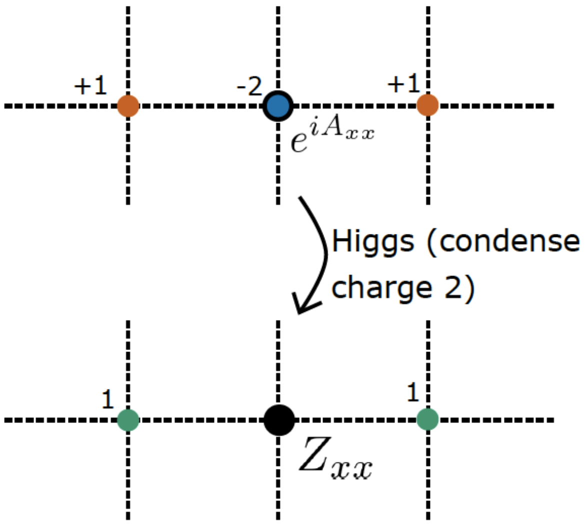
Study Higgs phases

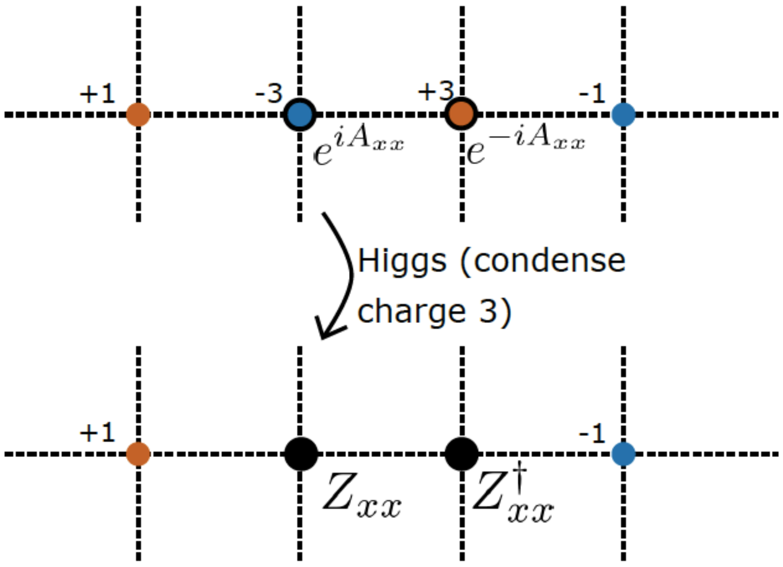
Condense charge p excitations, U(1) → Z_p

$$\mathcal{H} = E^2 + B^2 + \sum_{ij} U \cos(\partial_i \partial_j \theta + pA_{ij})$$

Take $U \to \infty$

Rank 2 scalar charge theory becomes “conventional” topological order upon Higgsing





Higgs procedure: derive Zp lattice model

$$\Delta_i \Delta_j \theta + p A_{ij} = 2\pi n$$

Pick gauge $\theta = 0$ using gauge transformation $\alpha(\mathbf{r}) = -\theta(\mathbf{r})/p$

$$A_{ij} = \frac{2\pi}{p} n \qquad e^{iA_{ij}} = Z_{ij} \qquad (-1)^{E_{ij}} = X_{ij}$$

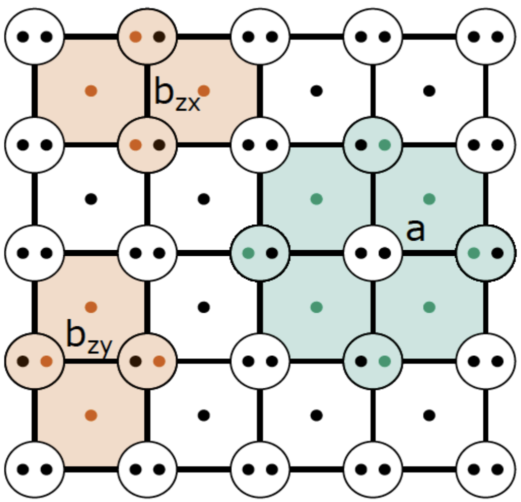
(p = 2)

$$-U \sum_{\mathbf{r}} a(\mathbf{r}) \equiv -U \sum_{\mathbf{r}} (-1)^{\sum_{i,j} \Delta_i \Delta_j E_{ij}}$$ Gauss law

$\cos B_{ij} \equiv b_{ij}$ → Magnetic terms = product of Z operators

E^2 terms → Zeeman terms $h \sum X_{ij}$

2D example



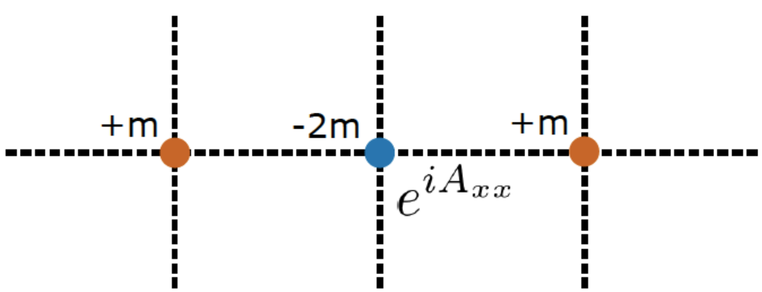
$$H_{2D} = -\frac{1}{g^2}\sum_{\text{links}}(b_{zx}+b_{zy}) - h_s \sum_{\text{sites},i} X_{ii} - h_p \sum_{\text{plaquettes}} X_{xy} - U\sum_{\text{sites}} a$$

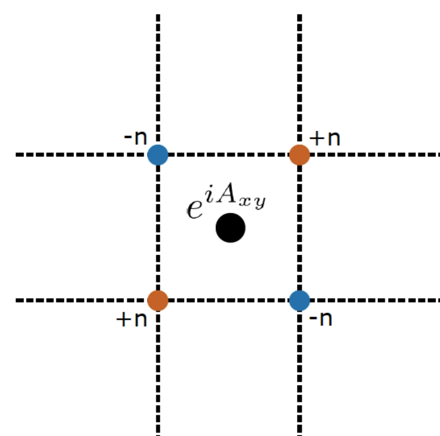
Generalize higher rank symmetric gauge theories subject to cubic rotational symmetry

Theory	Gauss' Law	Gauge transformation
(m,n) scalar	$m\sum_i \Delta_i^2 E_{ii} + n\sum_{i\neq j}\Delta_i\Delta_j E_{ij} = \rho$	$A_{ij} \to A_{ij} - \begin{cases} m\Delta_i^2\alpha & i=j \\ n\Delta_i\Delta_j\alpha & i\neq j \end{cases}$
(m,n) vector	$m\Delta_j E_{jj} + 2n\sum_{i\neq j}\Delta_i E_{ij} = \rho_j$	$A_{ij} \to A_{ij} - \begin{cases} m\Delta_i\alpha_i & i=j \\ n(\Delta_i\alpha_j + \Delta_j\alpha_i) & i\neq j \end{cases}$

If m/n is irrational, then cannot add B to Hamiltonian while respecting compactness of A (B ~ B + 2pi m ~ B + 2 pi n)

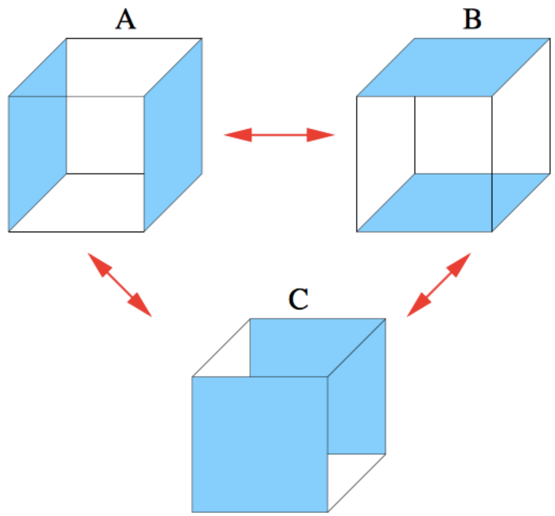
(m,n) scalar charge theory:





Note: (0,1) scalar charge theory is equivalent to resonating quantum plaquette model

Xu, Wu 2008
Pankov, Moessner, Sondhi 2007



$$
\begin{aligned}
H &= -t \sum_{\text{each cube}} \{|A\rangle\langle B| + |B\rangle\langle C| + |B\rangle\langle C| + h.c.\} \\
&+ V \sum_{\text{each cube}} \{|A\rangle\langle A| + |B\rangle\langle B| + |C\rangle\langle C|\}, \qquad (1)
\end{aligned}
$$

Possible relevance to SU(4) spins on cubic lattice

(0,1) scalar charge theory in (3+1)D is unstable to instantons

(m,n) scalar charge theory for m, n > 0 is probably stable

Two-dimensional models

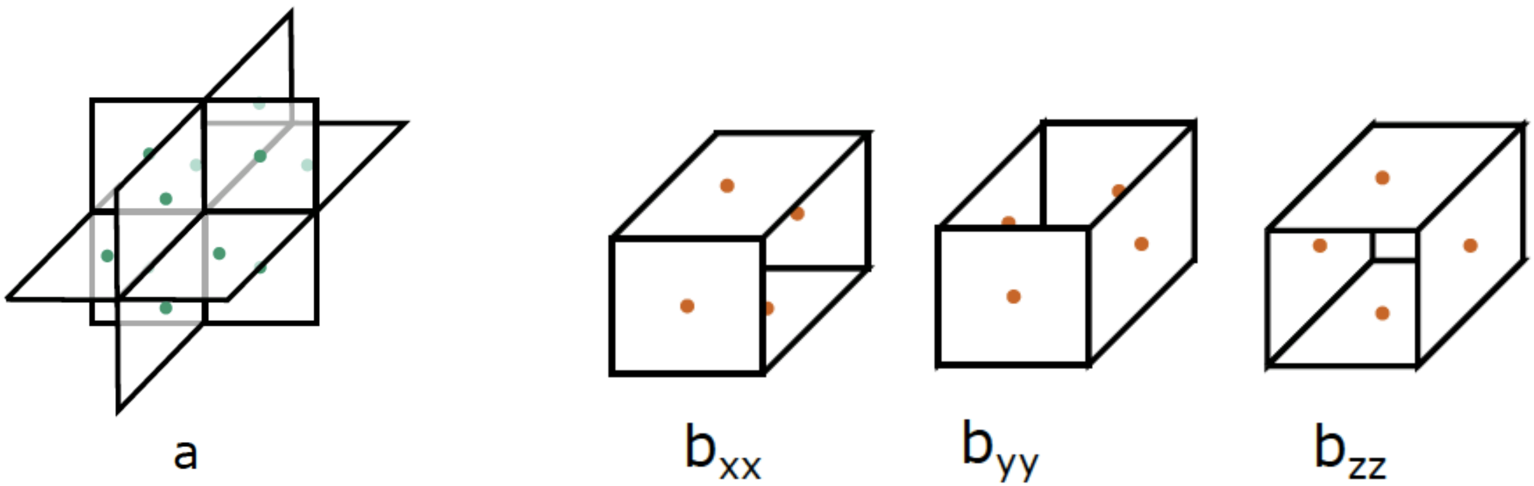
$U(1)$ **Charge Type**	(m,n)	**Higgs Phase**
$d=2$ scalar		
	$(2r+1,\ 2s+1)$	$\mathbb{Z}_2^3$ topological order
	$(2r, 2s+1)$	Trivial
	$(2r+1, 2s+2)$	Trivial
	$(1,0)$	$\mathbb{Z}_2^4$ topological order
$d=2$ vector		
	$(2r+1,\ 2s+1)$	$\mathbb{Z}_2^3$ topological order
	$(2r+2,\ 2s+1)$	$\mathbb{Z}_2^4$ topological order
	$(2r+1,\ 2s)$	Trivial
	$(0,1)$	Trivial

Three-dimensional models

$U(1)$ **Charge Type**	(m,n)	**Higgs Phase**
$d=3$ scalar		
	$(2r+1,\ 2s+1)$	$\mathbb{Z}_2^4$ topological order
	$(2r, 2s+1)$	X-Cube fracton order
	$(2r+1, 2s+2)$	Trivial
	$(1,0)$	$\mathbb{Z}_2^8$ topological order
$d=3$ vector		
	$(2r+1,\ 2s+1)$	$\mathbb{Z}_2^7$ topological order
	$(4r+2,\ 2s+1)$	$\mathbb{Z}_2$ topological order
	$(4r,\ 2s+1)$	Trivial
	$(2r+1,\ 2s)$	Trivial

(2r, 2s+1) scalar charge theory

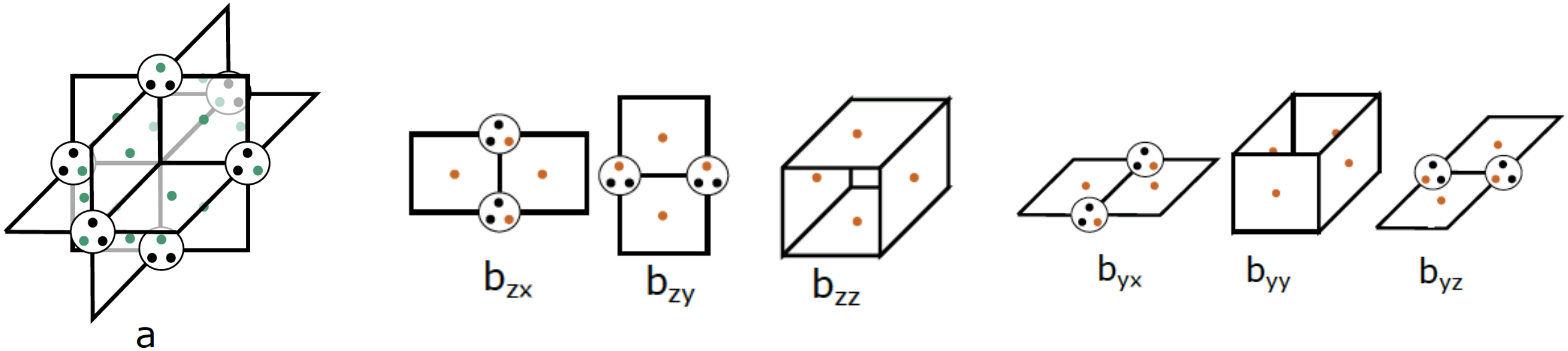
Higgs phase → X-cube model

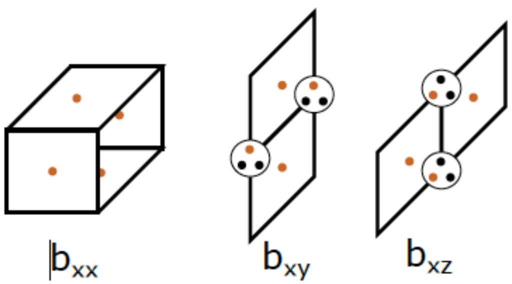


$$H = \sum a + \sum b_{ij}$$

For r > 0, this is a stable, gapless phase which transitions to a gapped fracton model

(2r+1, 2s+1) scalar charge Higgs theory





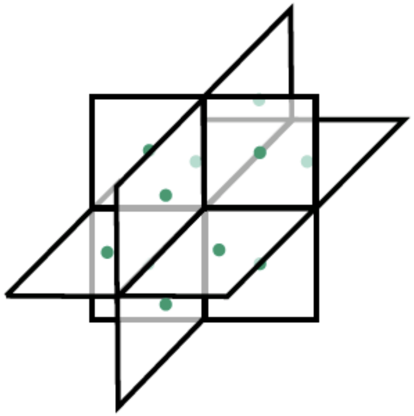
Applying Z_{ii} at each site creates two particles separated by two lattice spacings
→8 possible distinct types of e particles

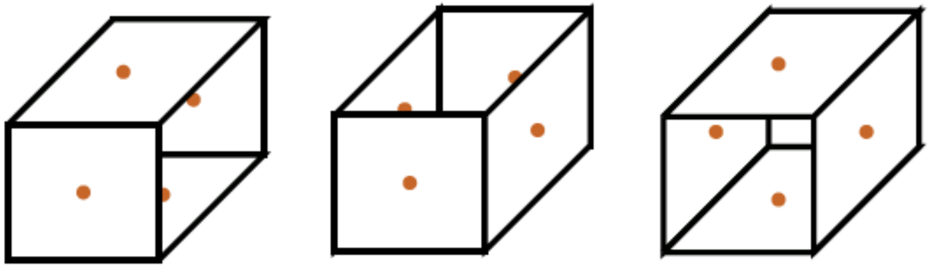
Applying Z_{ij} at each face creates 4 excitations (or convert 3 into 1)

→4 distinct types of e particles left → Z_2^4 topological order

Large h_f limit freezes out face spins → now we really have 8 distinct types of charges → Z_2^8 topological order

Large h_s limit → freeze out all site spins



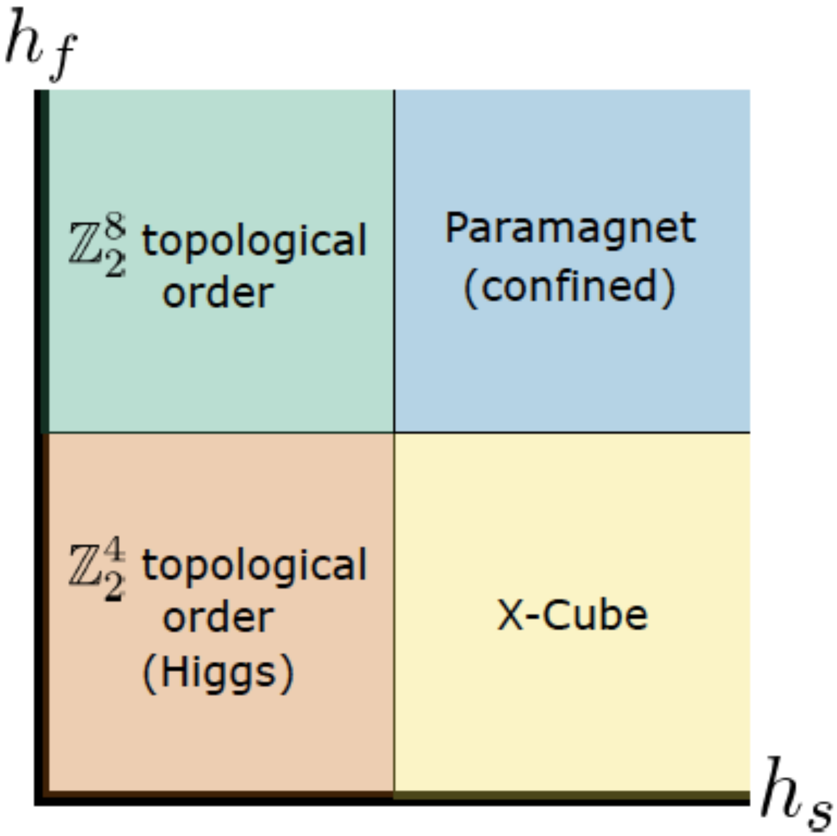


$$H = \sum a + \sum b_{ij}$$

X-cube model

Note: continuous rotational symmetry is completely incompatible with realizing the X-cube phase

Schematic phase diagram for (2r+1, 2s+1) scalar charge Higgs theory



Direct transition to X-cube theory?

Nature of transition?

U(1) higher rank symmetric gauge theories → constrained dynamics

Some of them Higgs to gapped fracton models
(e.g. (0,1) scalar charge theory)

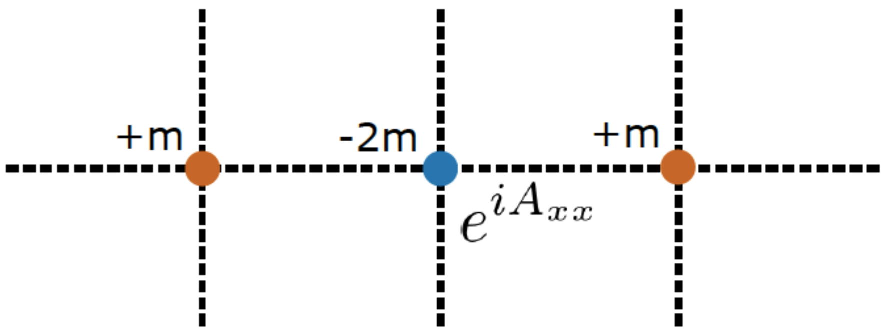
The others exhibit a certain constrained dynamics with no discrete Z_p analog. (e.g. (1,1) scalar charge theory)

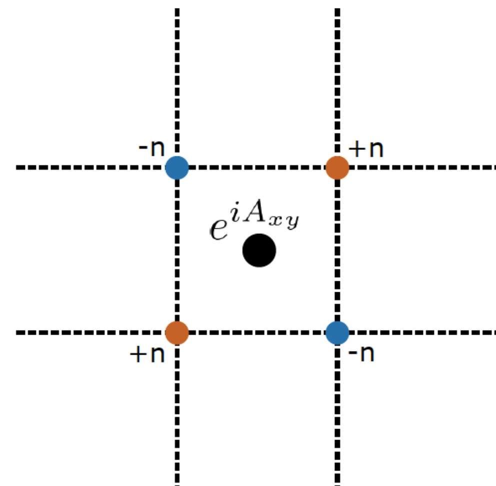
Generalization to other types of fracton models, like Haah's code?

Even more generalized U(1) gauge theories

In general, the Gauss law specifies the geometric configuration of charges created by local operators

$$m \sum_i \partial_i^2 E_{ii} + n \sum_{i \neq j} \partial_i \partial_j E_{ij} = \rho$$





Alternatively, given a geometrical configuration of charges, we can define a corresponding Gauss law

Even more generalized U(1) gauge theories

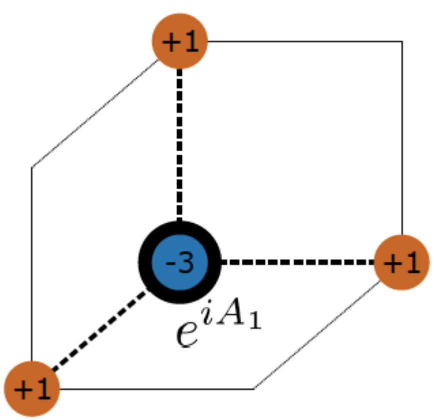
Consider a single pair of conjugate variables

$$[A_1(\mathbf{r}), E_1(\mathbf{r})] = i\delta^3(\mathbf{r} - \mathbf{r}')$$

Consider the Gauss law $$\rho(\mathbf{r}) = \sum_{i=1}^{3} \partial_i E_1(\mathbf{r})$$

Gauge transformation $$A_1 \to A_1 + \sum_i \partial_i \alpha$$

$$\phi \to \phi - \alpha$$



Lattice regularization:

$$\rho(\mathbf{r})/a_0 = -3E_1(\mathbf{r}) + \sum_{i=1}^{3} E_1(\mathbf{r} + a_0\hat{\mathbf{x}}_i)$$

Local operators create charges in shape of a pyramid

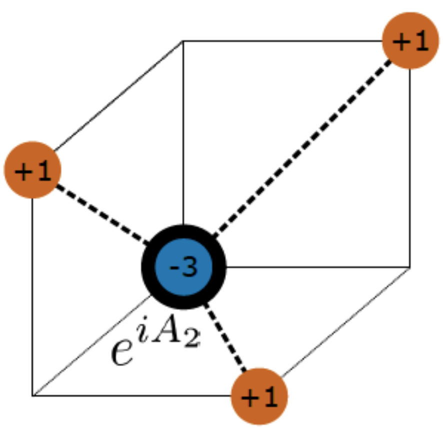
But, there is no gauge-invariant "magnetic field" that we can define.

Introduce another set:

$$[A_2(\mathbf{r}), E_2(\mathbf{r})] = i\delta^3(\mathbf{r} - \mathbf{r}')$$

$$D_2 E_2 = \rho \qquad D_2 = a_0 \sum_{i<j} \partial_i \partial_j - 2 \sum_i \partial_i$$

$$A_2 \to A_2 + (a_0 \sum_{i<j} \partial_i \partial_j + 2 \sum_i \partial_i)\alpha$$



$$\rho(\mathbf{r})/a_0 = -3E_2(\mathbf{r}) + \sum_{i<j} E_2(\mathbf{r} + a_0(\hat{x}_i + \hat{x}_j))$$

Combine the two allowed charge configurations:

Gauss law $$\rho = D_1 E_1 + D_2 E_2$$

$$D_1 = \sum_i \partial_i, \quad D_2 = a_0 \sum_{i<j} \partial_i \partial_j - 2 \sum_i \partial_i$$

$$A_l \to A_l - \tilde{D}_l \alpha \qquad \phi \to \phi - \alpha$$

$$\tilde{D}_1 = -\sum_i \partial_i \qquad \tilde{D}_2 = a_0 \sum_{i<j} \partial_i \partial_j + 2 \sum_i \partial_i$$

Now we have a gauge-invariant magnetic field $$B = \tilde{D}_1 A_2 - \tilde{D}_2 A_1$$

Gauss law $\rho = D_1 E_1 + D_2 E_2$

$$D_1 = \sum_i \partial_i, \quad D_2 = a_0 \sum_{i<j} \partial_i \partial_j - 2 \sum_i \partial_i$$

$$B = \tilde{D}_1 A_2 - \tilde{D}_2 A_1$$

Maxwell theory $\mathcal{H}_{\rm G} = \sum_i E_i^2 + \frac{1}{2g^2} B^2$

$$\omega^2 = (a_0)^2 \left(\sum_{i<j} k_i k_j\right)^2 + 5 \left(\sum_i k_i\right)^2$$

Theory has SO(2) rotational symmetry about (111) axis

This theory has infinitely many conserved quantities

Take u = (111) direction

$$Q_f \equiv \int du dv dw f(v,w) \rho(u,v,w) = 0$$

For any harmonic function:

$$(\partial_v^2 + \partial_w^2) f(v,w) = 0$$

Can couple the theory to charge p matter:

$$\mathcal{H} = \mathcal{H}_{\mathrm{G}} + \mathcal{H}_{\mathrm{M}}$$

$$\mathcal{H}_M = \frac{L^2}{2M} - \sum_{i=1,2} V_i \cos(\tilde{D}_i \phi - pA_i)$$

where $$D_1 E_1 + D_2 E_2 = pL$$

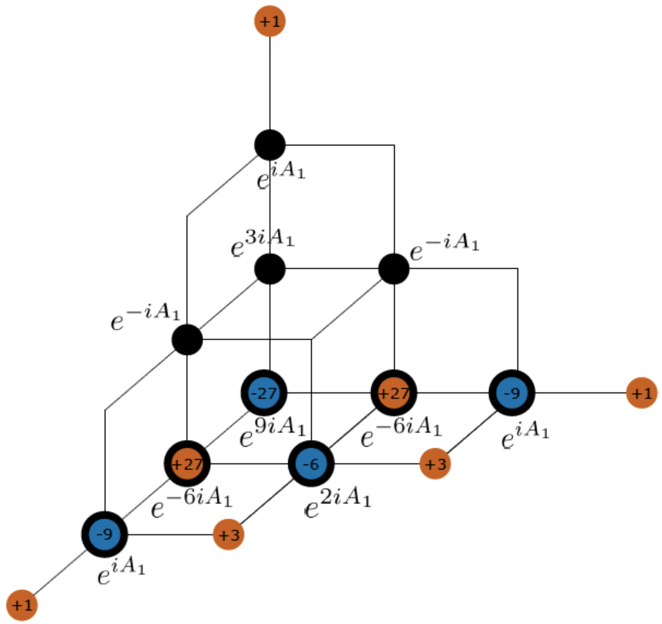
Discretize on a lattice

$$H_M = \sum_{\mathbf{r}} V_1\, b_{\mathbf{r}}^3 b_{\mathbf{r}+\hat{x}}^\dagger b_{\mathbf{r}+\hat{y}}^\dagger b_{\mathbf{r}+\hat{z}}^\dagger e^{iA_1} + V_2\, b_{\mathbf{r}}^3 b_{\mathbf{r}+\hat{x}+\hat{y}}^\dagger b_{\mathbf{r}+\hat{x}+\hat{z}}^\dagger b_{\mathbf{r}+\hat{y}+\hat{z}}^\dagger e^{iA_2} + H.c.$$

Gauge non-invariant hopping terms are forbidden because they take the system out of the low energy subspace defined by the gauge constraint

Exponential geometrical confinement of charges

Charges can be separated by repeated application of exp(i A) operators



$$E \sim e^{R/a_0}$$

Lattice is still important – charges only appear at lattice scale

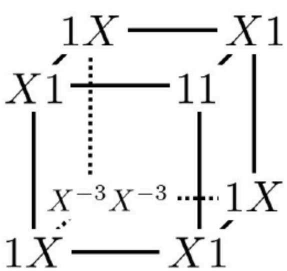
Taking $a_0 \rightarrow 0$ in continuum Gauss law, then $A_2 + 2\,A_1$ would be gauge-invariant, which is incorrect at lattice scale

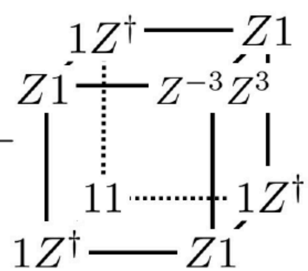
Haah 2018
D. Bulmash MB 2018

$$\mathcal{H}_M = \frac{L^2}{2M} - \sum_{i=1,2} V_i \cos(\tilde{D}_i\phi - pA_i)$$

Condense charge p matter by taking $V \to \infty$

If we discretize on a cubic lattice:

$H = -\sum_{\text{cubes}}$ 

$+$ 

$-h \sum_{\text{spins}} X + \text{h.c.}$

Z_p version of Haah's cubic code

$$\mathcal{H} = \mathcal{H}_{\mathrm{G}} + \mathcal{H}_{\mathrm{M}} \qquad \mathcal{H}_{\mathrm{M}} \approx \frac{1}{2}\sum_i V_i(\tilde{D}_i\phi - pA_i)^2$$

Effective field theory for Haah's code

$$\mathcal{H}_{\mathrm{G}} = \sum_i E_i^2 + \frac{1}{2g^2}B^2$$

General construction

Geometric configuration of charges created by local operators ⟷ Gauss law

$$\sum_{l=1}^{N} D_l^a E_l(\mathbf{r} = \rho_a(\mathbf{r})$$

N geometric charge configurations

M charge flavors : a = 1,..., M

$$A_l \to A_l - \tilde{D}_l^a \alpha^a \qquad \theta_a \to \theta_a - \alpha^a$$

"Magnetic field" $B^k = \sum_l C_l^k A_l$ where $\sum_l C_l^k \tilde{D}_l^a = 0$

When N > M, expect that it is always possible to define gauge-invariant magnetic field (Proof for M = 1, 2)

Example: For M = 1, N = 2, $B = \tilde{D}_1 A_2 - \tilde{D}_2 A_1$

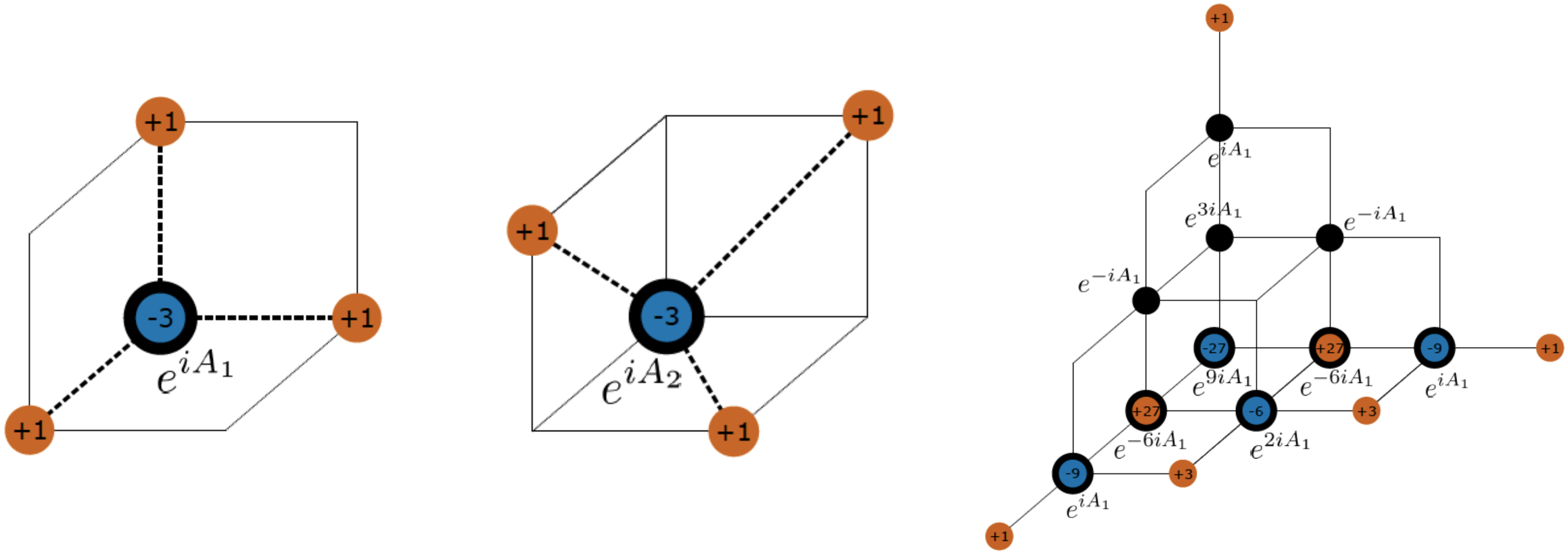
$$\mathcal{L} = \frac{1}{2}\sum_i \left(\sum_a \tilde{D}_i^a A_0^a + \partial_t A_i\right)^2 - \frac{1}{2}\sum_k (B^k)^2 +$$

$$+ \sum_{ik} \theta_{ik} (\sum_a \tilde{D}_i^a A_0^a + \partial_t A_i) B_k$$

Mobility of charges = tiling problem

(im)mobility of charges derives from properties of tiling of the basic set of geometric charge configurations defined by Gauss laws

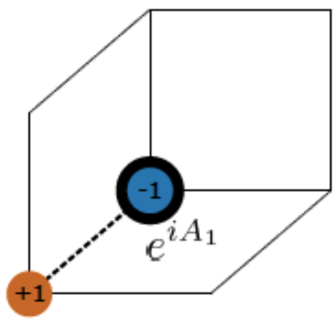
The two pyramids defining U(1) Haah code theory can never be tiled together to give only two far-separated charges

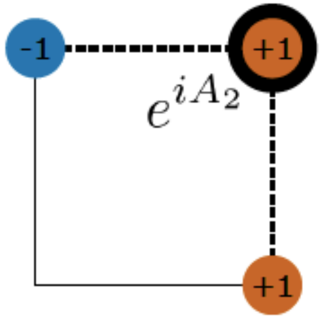


Another example: Sierpinski prism model

Consider the following charge configurations

Yoshida 2013
D. Bulmash, MB 2018





$$D_1 E_1 + D_2 E_2 = \rho$$

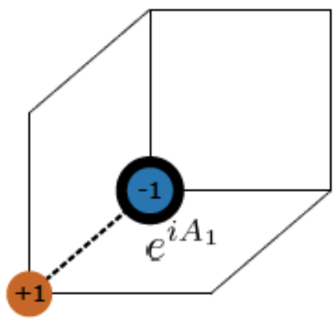
No global charge conservation

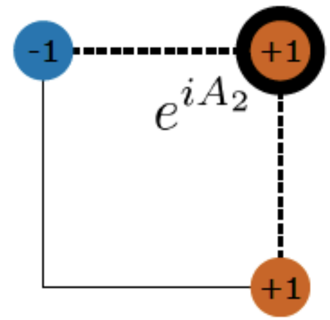
$$D_1 = \partial_z \qquad D_2 = a_0 \partial_x \partial_y + \partial_y + a_0^{-1}$$

$$B = \tilde{D}_2 A_1 - \tilde{D}_1 A_2$$

In-plane dynamics is fractal. Charges are mobile in z-direction

Another example: Sierpinski prism model





$$D_1 E_1 + D_2 E_2 = \rho$$

No global charge conservation

$$D_1 = \partial_z \qquad D_2 = a_0 \partial_x \partial_y + \partial_y + a_0^{-1}$$

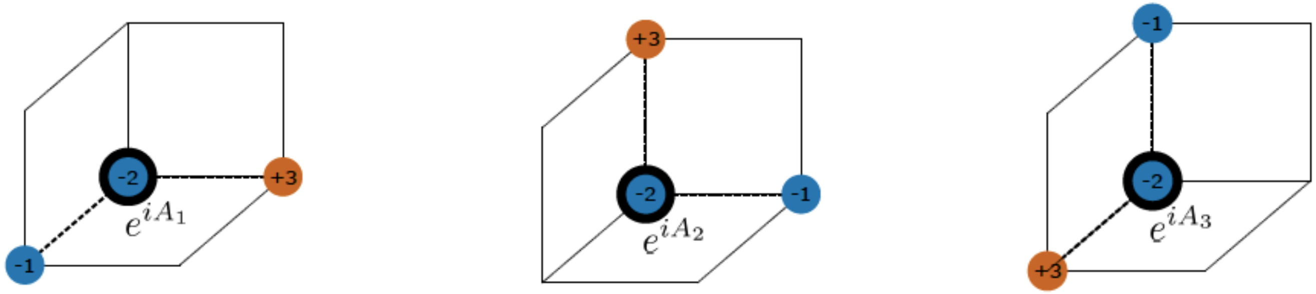
$$\mathcal{H} = E^2 + B^2$$

$$\omega^2 = a_0^{-2} + \left(k_y^2 - 2k_x k_y + k_z^2\right) + a_0^2 k_x^2 k_y^2$$

$$\Delta_{min}(k_x) = \frac{1}{a_0^2} \frac{1}{1+a_0^2 k_x^2}$$

low-energy physics occurs at lattice scale

U(1) model with no Z_p counterpart



$$\rho = (3\partial_x - \partial_z)E_1 + (3\partial_y - \partial_x)E_2 + (3\partial_z - \partial_y)E_3$$

$$B_i = \epsilon_{ijk}\tilde{D}_j A_k$$

Discretize Gauss law on cubic lattice:

$$\rho(\mathbf{r})/a_0 = \sum [3E_i(\mathbf{r} + a_0\hat{\mathbf{x}}_i) - 2E_i(\mathbf{r})] - E_1(\mathbf{r} + a_0\hat{\mathbf{z}}) - E_2(\mathbf{r} + a_0\hat{\mathbf{x}}) - E_3(\mathbf{r} + a_0\hat{\mathbf{y}})$$

Charges become mobile in all three directions upon breaking U(1) gauge symmetry down to Z_p

Summary and outlook

- Large class of generalized U(1) gauge field theories. Defined by specifying geometric configurations of charge configurations.

  N geometric charge configurations → N components of electric/gauge fields

  Generally N > M sufficient for existence of gauge-invariant magnetic field.

- Mobility of charge configurations = tiling problem for the geometric shapes

- Many interesting phase diagrams. Direct transition from Z_2^4 to X-cube?

- Examples of stable gapless U(1) theories that Higgs to X-cube model

- Examples of non-trivial U(1) theories (either "type I" or "type II") with no discrete Z_p analog

- How to describe other gapped fracton models from this perspective?
  - Chamon model; "Twisted" fracton models;
  - Non-Abelian fracton models obtained through layer constructions